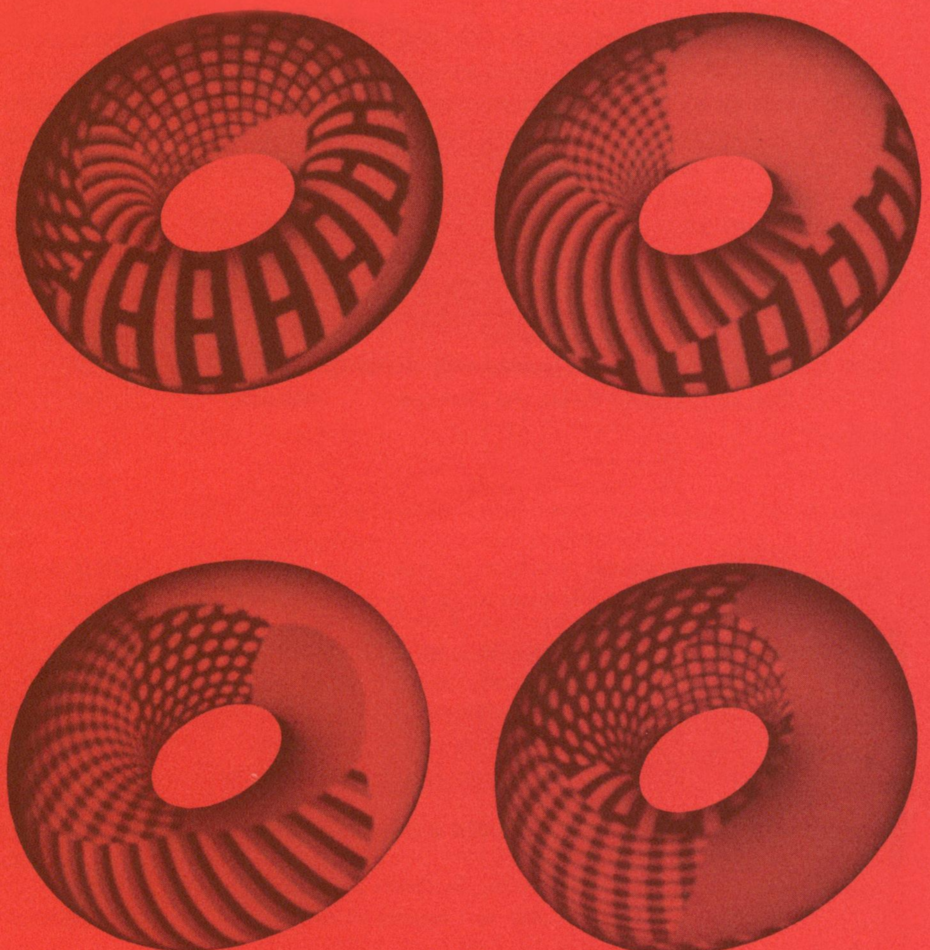
MATHEMATICS MAGAZINE



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MAP-COLORING TORI, ETC. • SYMMETRIC KNOTS
JUST ONE LOCAL EXTREMUM • IMITATING 3X + 1

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COVER: Torus with complete 7-map. See pp. 131–145 and pp. 159–160.
Design by Charles Gunn.

EDITORIAL POLICY

Mathematics Magazine is a journal which aims to provide inviting, informal mathematical exposition. Manuscripts for the *Magazine* should be written in a clear and lively expository style and stocked with appropriate examples and graphics. Our advice to authors is: say something new in an appealing way or say something old in a refreshing way. The *Magazine* is not a research journal and so the style, quality, and level of articles should realistically permit their use to supplement undergraduate courses. The editor invites manuscripts that provide insight into the history and application of mathematics, that point out interrelationships between several branches of mathematics and that illustrate the fun of doing mathematics.

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AUTHORS

Saul Stahl ("The Other Map Coloring Theorem") received his Ph.D. in mathematics from Western Michigan University in 1975. He taught for two years at Wright State University and then moved on to the University of Kansas, where he is now an associate professor. He classifies himself as a combinatorialist and one of his main interests is the combinatorial underpinning of the Riemann surfaces. The history of mathematics is one of his hobbies and he believes that the development of many mathematical problems and their solutions makes for enjoyable and edifying reading even when most of the details of their proofs are omitted.

ILLUSTRATIONS

Mobius Strip II, a woodcut by M. C. Escher, p. 142, is from the collection of **C. V. S. Roosevelt.**

Hans Reddinger produced the graphics on p. 150.

All other illustrations were provided by the authors.

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The Other Map Coloring Theorem

The problem of coloring a map on a pretzel or Möbius band was solved before the Four-Color Problem.

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The Four Color Problem has probably been the most notorious mathematical problem of modern times. This problem asks whether four colors suffice to color every planar map so that adjacent countries, i.e., countries that share a border of positive length, receive different colors. This question's deceptive simplicity attracted many would-be solvers who oft times spent years on their search for a solution. Most returned empty-handed, or worse, with a false proof. Some were fortunate enough to have devised a new twist on the original problem that was sufficiently interesting to attract the attention of other aficionados. The **Heawood Conjecture**, one of the earliest of these offshoots, proved to be also one of the most fascinating and difficult. This other coloring conjecture guessed at the number of colors required by maps on other, more complicated, surfaces. Surprisingly enough, even though this later problem seems more difficult than its planar progenitor, Heawood's conjecture was actually verified a decade earlier. In my opinion this verification marks a milestone in the development of the modern combinatorial approach to geometry. It is my intention here to formulate this problem, recount its history, and discuss its relationship to the original Four Color Problem, as well as to other branches of mathematics.

Heawood's Conjecture

It is generally agreed that the Four Color Conjecture was first formulated by Francis Guthrie, a graduate student at University College, London, in 1852. Appel and Haken's proof of this conjecture is described by them in [**1**] and has also received wide publicity elsewhere, so I will not discuss it here and only refer to it when it provides an interesting parallel or contrast with the other map coloring theorem. The first false proof of the Four Color Conjecture to be published was given in 1879 by A. B. Kempe [**14**], barrister and part-time mathematician. The validity of this proof was not challenged until 1890 when P. J. Heawood published a paper [**10**] in which he accomplished several things. First, he pointed out the error in Kempe's reasoning. Next, he salvaged the remains of Kempe's fallacious proof by using its techniques to show that every planar map can be colored with *five* colors. Finally, he went on to state and solve, so he thought, the same problem in a new context. This came to be known as the Heawood Conjecture.

Before describing this new context, a minor clarification is in order. Whenever a map is mentioned above and in the sequel, it is to be understood that each country forms a single contiguous geographical unit, unlike pre-Bangladesh Pakistan. Also, in order to minimize redundancy it will be assumed that whenever a map is colored, the coloring satisfies this map-coloring constraint: adjacent countries receive different colors.

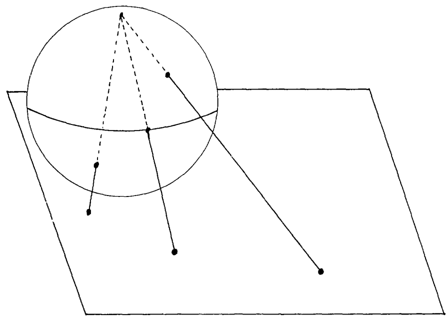


FIGURE 1. **Stereographic projection. A technique used to represent spherical maps on flat paper. The particular version portrayed here will represent regions near the south pole fairly faithfully, but will greatly distort northern countries. Nevertheless, it does preserve the adjacency pattern.**

The plane is not the only surface on which maps can be drawn; they can of course also be drawn on the surface of a sphere. However, coloring maps on spheres is really not different from coloring them on the plane. The well-known stereographic projection of FIGURE 1, often employed by map makers, can be used to convert any spherical map to a planar one, and the coloring pattern of the planar projection of a spherical map can also be used to color the original map on the sphere. Nothing is therefore to be gained by reformulating the Four Color Problem for the sphere. Maps on the surface of a cone or a pyramid do not provide any new challenges either, for these surfaces can be easily deformed into spherical ones in a manner that preserves the adjacency pattern of any maps drawn on them. So, if one is to find a surface that will yield a genuinely new coloring problem, then this surface cannot be deformable into a sphere in any "nice" way. One such surface is a torus—the surface of the doughnut. In M_5 (FIGURE 2) we have a toroidal map consisting of five countries every two of which are adjacent. Such a map clearly requires *five* colors. In fact, M_6 and M_7 (FIGURES 3, 4) are toroidal maps with six and seven countries in which every two countries are adjacent to each other. Consequently, these maps require six and seven colors, respectively. The subsequent discussion centers around maps of this type and it will therefore be convenient to have a name for them. Accordingly, a **complete m-map** is a map which consists of m countries every two of which are adjacent. Such a map clearly requires m colors. It is natural to ask at this point whether there exists a complete 8-map. The answer is negative for the torus.

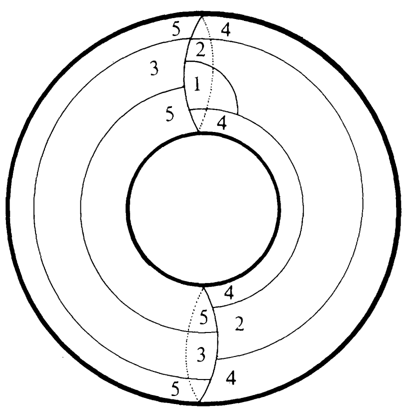


FIGURE 2. **M_5, a toroidal map of five countries every two of which touch each other.**

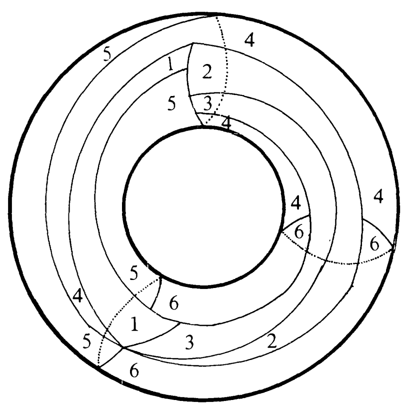


FIGURE 3. **M_6, a toroidal map of six countries every two of which touch each other.**

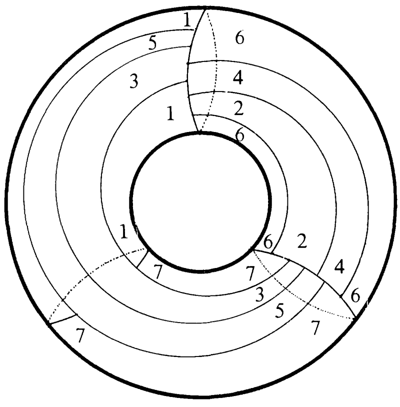


FIGURE 4. M_7, **a toroidal map of seven countries every two of which are adjacent.**

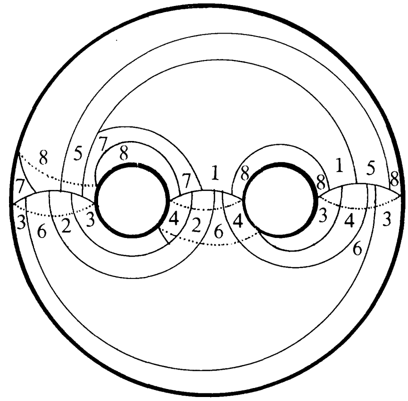


FIGURE 5. M_8, **a complete 8-map on the double torus** S_2.

The mathematician A. F. Möbius (see [2]) had already pointed out in the 1840's that it is not possible to draw a complete 5-map in the plane. In later years this observation was sometimes misconstrued as a solution of the Four Color Problem. It is, of course, no such thing. The nonexistence of a complete 5-map in the plane does not preclude the existence of an incomplete planar map with a large number of countries, whose complexity actually requires five or more colors. We now have an analogous situation on the torus, for it supports a complete 7-map and no complete 8-map can be drawn on it. Does this mean that every toroidal map can be colored with seven colors? Such indeed is the case. What is surprising is the relative ease with which this answer was obtained. So simple is this proof, that its main points deserve to be brought out here.

In any map, let n denote the number of its **nodes**, namely, the number of points at which three or more countries meet. Let b denote the number of **borderlines** in the map, in other words, curves that go from one node to another. The Euler-Poincaré formula [15], one of the central facts of geometry, implies that if a toroidal map has m countries, then

$$n - b + m = 0. \tag{1}$$

For the maps M_5, M_6, and M_7, this formula takes the forms $9 - 14 + 5 = 0$, $9 - 15 + 6 = 0$, and $14 - 21 + 7 = 0$, respectively. For the sake of accuracy let me point out that the Euler-Poincaré formula does not apply to maps in which one country has the shape of a ring and completely surrounds one or several other countries. Such maps, however, can be easily disposed of by certain standard reduction techniques and will be ignored in the sequel.

Let A denote the average number of borderlines that occur on the boundaries of the m countries of a given map. Then mA is the count of all the occurrences of borderlines on the boundaries of these countries. Since each of the b borderlines occurs on 2 of these boundaries, it follows that

$$b = \frac{mA}{2}. \tag{2}$$

Turning our attention to the nodes, observe that A must also be the average number of *nodes* on the boundaries of these countries, since each country has an equal number of nodes and borderlines on its boundary. However, in contrast with the borderlines, we cannot specify the number of countries on whose boundary a given node will occur. Still we can say that each node appears on the boundaries of at least 3 countries. This gives us the following weak analog of equation (2), which is nevertheless sufficient for our purpose:

$$n \leqslant \frac{mA}{3}. \tag{3}$$

Combining (1), (2), and (3) we obtain

$$\frac{mA}{3}-\frac{mA}{2}+m\geqslant n-b+m=0,$$

$$m\left(1-\frac{A}{6}\right)\geqslant 0.$$

Since m is a positive quantity, we may conclude that $A\leqslant 6$. In other words, the average number of borderlines on the boundary of a typical country in any toroidal map does not exceed 6. Consequently we obtain the following surprising fact: *in every toroidal map there must be a country that is adjacent to no more than six other countries*. This fact immediately points out the impossibility of drawing a complete 8-map on the torus, since in such a map every country is necessarily adjacent to *seven* other countries. This fact can also be used to define an algorithm for coloring any toroidal map with $6+1=7$ colors. Thus, every map on the torus can be colored with seven colors and some such map (the complete 7-map) actually requires seven colors. In other words, the answer to the toroidal analog of the Four Color Problem is that 7 colors suffice. Heawood was the first to formulate this analog clearly and to answer it. He did much more, however.

Why does the torus allow for more complicated maps than the sphere? (The complexity of a map is here equated with the number of colors it requires.) One might say that this additional complexity is made possible by the hole in the torus—the doughnut hole. The adjacency pattern of a spherical map is constrained by the fact that a country that lies completely in the northern hemisphere cannot possibly be adjacent to one that lies entirely in the southern hemisphere. However, if a tunnel is bored from the north pole of the sphere to its south pole, thus converting it essentially into a doughnut, it now becomes possible for a northern country to reach out and touch a southern one *without crossing the equator*, namely, along the walls of the tunnel. From this point of view, a tunnel through a surface may be thought of as a bridge that connects two parts of the surface and allows for a higher degree of complexity in the adjacency pattern of its maps. So if we wish to find a surface that supports a complete 8-map (which the torus does not), it is clear what must be done. A tunnel should be bored through the torus, or, equivalently, two nonintersecting tunnels should be bored through the sphere. The resulting surface, the double torus, does indeed support the complete 8-map M_8 of FIGURE 5.

We now have a technique for creating surfaces that would seem to allow for maps requiring arbitrarily many colors. All that needs to be done is to bore enough tunnels through the sphere. We will call the surface that results from boring g nonintersecting tunnels through the sphere the **surface of genus** g and denote it by S_g. Thus S_2 is the double torus, the torus itself is S_1, and the surface of the unperforated sphere is S_0. The great mathematician Bernard Riemann brought these surfaces into the foreground of mathematics in 1851 [**18**] when he showed that they play a focal role in the calculus of complex variables. His theory was so central to nineteenth century mathematics that it has been said [**4**, p. 121] that at one time all research mathematicians had to be familiar with it. It is therefore not surprising that shortly after the Four Color Problem was formulated for planar maps, mathematicians would ask the same question in the context of Riemann's surfaces.

The Euler-Poincaré formula states that whenever a map with m countries, b borderlines, and n nodes is drawn on the surface S_g, then these parameters are linked by the equation

$$n-b+m=2-2g. \qquad (4)$$

Heawood used this fact to show that if $g\geqslant 1$, *then any such map can be colored with no more than*

$$H_g=\left[\frac{1}{2}\left(7+\sqrt{1+48g}\right)\right] \qquad (5)$$

colors, where the brackets in (5) denote the integer part of the enclosed number. This guarantees that any map on the torus can be colored with $H_1=[\frac{1}{2}(7+\sqrt{1+48}\,)]=7$ colors, that every map on the double torus can be colored with $H_2=[\frac{1}{2}(7+\sqrt{1+96}\,)]=[8.42\ldots]=8$ colors, and that

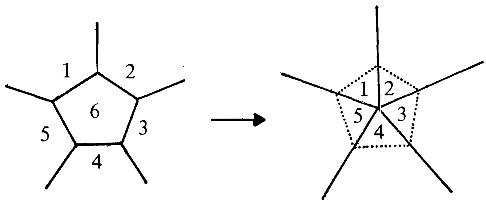


FIGURE 6. **A reduction process for maps that diminishes the number of countries without increasing the number of required colors.**

every map on S_3 can be colored with 9 colors.

Heawood's proof that the number of colors specified in (5) is actually sufficient is paraphrased as follows. Call those borderlines along which a country touches its neighbors its **contacts** (clearly every country possesses at least as many contacts as neighbors, and sometimes more). Suppose that there is an integer x such that every map on S_g has a country with fewer than x contacts. Then we claim that x colors suffice to color every map on S_g. For in any such map we can annihilate the country with less than x contacts, making those round it close up in the space which it occupied (FIGURE 6), and obviously the original map can be done in x colors if the reduced map can (for whatever the coloring of the countries around, there would be a color to spare for the annihilated country). Having thus described an induction process, Heawood now needed only to find the smallest x that satisfied this condition, and he used an averaging argument that is essentially the same as the one used above to argue that the torus does not support a complete 8-map. First, however, Heawood observed that every map on any surface can be converted by the operations described in FIGURE 7 to a map in which every node appears on the boundary of *exactly* 3 countries, this new map requiring no more colors than the original one. Consequently, with n, b, m, A denoting the same quantities as before, equation (2) still holds, whereas inequality (3) can now be replaced by the equation

$$n = \frac{mA}{3}. \tag{6}$$

If we now substitute (2) and (6) in (4) we obtain

$$A = 6\left(1 + \frac{2g-2}{m}\right). \tag{7}$$

As was argued for the torus, every map on S_g clearly has a country with no more than A adjacencies, so any x satisfying

$$x > A \tag{8}$$

should do. Expression (7), however, still contains an m which depends on the map rather than the surface. To eliminate this m Heawood observed that A diminishes as m increases, and since every map with no more than x countries clearly must have a country with fewer than x contacts, it follows that in order for (8) to hold we need only guarantee that

$$x > 6\left(1 + \frac{2g-2}{x+1}\right). \tag{9}$$

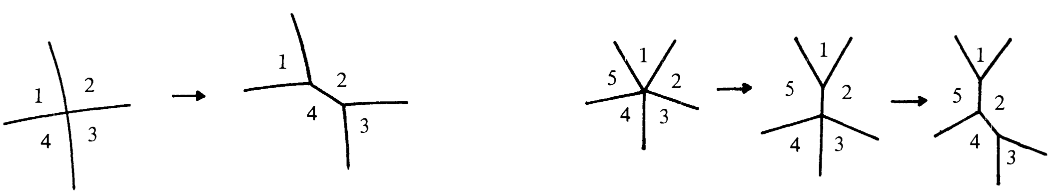


FIGURE 7. **How to transform any map to one requiring no more colors, but in which every node is on the boundary of exactly three countries.**

Thus, if we let H_g denote the smallest of all the integers x that satisfy inequality (9), then the maps on the surface S_g can be colored with H_g colors. Since for all smaller values of x the reverse inequality must hold, this can be rephrased as saying that H_g is the largest integer x that satisfies the inequality

$$x-1 \leqslant 6\left(1+\frac{2g-2}{x}\right).$$

This is a straightforward quadratic inequality in x, whose solution yields the value specified in (5).

The above argument is essentially the same as the one presented by Heawood himself. I have even gone so far as to leave intact some of the rougher edges and omissions in his proof, partly for the sake of brevity, and partly to prepare the reader for Heawood's real mistake, which is to be discussed shortly. Cleaner arguments can be found in [**2**] and [**20**].

To demonstrate that H_g colors are in fact required by some map on S_g, Heawood also drew a complete 7-map on the torus—essentially the same as M_7—but only *asserted* the existence of a complete H_g-map on the surface S_g for all $g \geqslant 2$. "For more highly connected surfaces," Heawood stated, "it will be observed that there are generally contacts enough and to spare for the above number of divisions [countries] each to touch each." In other words, Heawood thought it was easy to see that a complete 9-map could be drawn on S_3. Anyone who tries to draw such a map will quickly discover that not only is this quite a difficult task, but it also sheds no light on the question of how to draw a complete 10-map on S_4. Thus, what Heawood accomplished was to show that no map on the surface of genus $g \geqslant 1$ requires more than H_g colors. What he failed to show was that a map requiring this number of colors could actually be drawn on S_g. So a question he believed he had both raised and answered, in fact remained open. The number H_g defined in (5) acquired the name **Heawood number** and the assertion that *The Heawood number H_g is the largest number of colors required to color any map on the surface of genus $g \geqslant 1$* came to be known as the **Heawood Conjecture**.

The observant reader will have noted the qualification $g \geqslant 1$ that occurs in the statement of Heawood's conjecture. Were this qualification absent, the conjecture would apply to the surface S_0 (the sphere), and it would assert that any spherical map could be colored with no more than $H_0 = [\frac{1}{2}(7+\sqrt{1+0})] = 4$ colors! In other words, the removal of the inequality $g \geqslant 1$ would make the Heawood Conjecture contain the Four Color Conjecture as a special case. Unfortunately, the condition $g \geqslant 1$ cannot be disregarded. In Heawood's proof, the quantity

$$A = 6\left(1+\frac{2g-2}{m}\right)$$

was observed to be a *decreasing* function of m. This of course fails to be the case when $g = 0$, and so the proof breaks down for this value of g. This is a good example of the many near misses that proliferate in the history of the Four Color Problem.

Heffter's contributions

The deficiency in Heawood's paper was pointed out one year after its publication by Lothar Heffter [**11**]. Heffter also realized that the problem needed a new approach. Trying to draw complicated maps on 2-dimensional drawings of perforated spheres was a very cumbersome way to attack this problem. As an alternative he suggested that the adjacency pattern of a map (which, after all, is all that matters here) be recorded in the following manner. Suppose an inhabitant of some country were to inspect its borders by traveling along them in, say, the counterclockwise direction (counterclockwise from the point of view of an observer stationed right above his country). This inhabitant might then record, in order, the neighboring countries along whose border he is traveling. For instance, in the map M_4 of FIGURE 8, the inspector of country 1, if he starts his tour from a location on the border his country shares with country 2, would write 2, 4, 3 in his logbook. Had he started from a location on the border his country shares with country 3, his entry would have been 3, 2, 4. These two records are considered to be the same since it is only the cyclic ordering of the neighbors that matters to us. Now, country 2's inspector, in touring his

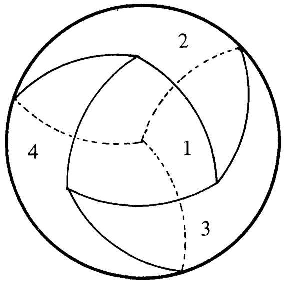


FIGURE 8. M_4, **a complete 4-map on the sphere.**

country	adjacency record		
1)	2	4	3
2)	1	3	4
3)	4	2	1
4)	3	1	2

FIGURE 9. A_4, **the adjacency pattern for the map** M_4.

borders, would write 1, 3, 4. The information obtained from these tours and those of the inspectors of countries 3 and 4, can be tabulated as the array A_4 in FIGURE 9. When applied to the toroidal complete 7-map M_7, this process yields the array A_7 in FIGURE 10. The array A_7 has a very exciting pattern. Every row can be derived from the previous one by the addition of 1 to each entry of the latter. This arithmetic is modulo 7; that is, we stipulate that $7+1=1$ and that the first row follows the seventh one.

country	adjacency record					
1)	2	4	3	7	5	6
2)	3	5	4	1	6	7
3)	4	6	5	2	7	1
4)	5	7	6	3	1	2
5)	6	1	7	4	2	3
6)	7	2	1	5	3	4
7)	1	3	2	6	4	5

FIGURE 10. A_7, **the adjacency pattern for the map** M_7 **in Figure 4.**

Before we go on to discuss these arrays in general, it should be pointed out that two miracles occurred in the passage from the complete 7-map M_7 to its array A_7. In the first place, the array displays a pattern, or a symmetry, that is totally obscured in the original map. Symmetry, of course, is one of the strongest tools of mathematics and of science. Hence, finding it in such an unexpected place is indeed an undisguised blessing. The second observation is even more surprising. The numbers $1,2,3,\ldots$ were used as a matter of convenience only. We could, and perhaps should, have used labels such as "Spain" or "Union of the Free Toroidal Republics." Nevertheless, we find that these countries, when symbolized by numbers, follow a very rigid arithmetical pattern. This phenomenon has been observed and exploited in other coloring problems, but *not* in the proof of the Four Color Theorem. Both of these miracles were crucial to the eventual resolution of Heawood's conjecture.

Let us now return to the arrays themselves. The array A_4 obtained from the complete 4-map M_4 does not quite conform to the nice pattern that was discovered in A_7. It seems that in certain columns one might need to subtract rather than add 1. Still, there is enough regularity in the rows of this array to justify some hope for a general pattern.

An obvious question at this point is: *which arrays correspond to some complete m-map*? Such an array must clearly have m rows, and the ith row must be some permutation of the numbers $1,2,\ldots,i-1,i+1,\ldots,m$. Let us call this requirement the **shape constraint**. An example of another array that satisfies the shape constraint is B_4, shown in FIGURE 11. A little experimentation, however, will quickly convince the reader that the array B_4 cannot correspond to any complete 4-map. This can be reasoned out by observing that according to the first row of this array, country 1's inspector, traveling counterclockwise, encounters country 2 just before country 3. Hence, the node formed by countries 1, 2, and 3 must form the configuration of FIGURE 12. However, according to this portion of the map, country 2's inspector must encounter country 3

country	adjacency record		
1)	2	3	4
2)	3	4	1
3)	4	1	2
4)	1	2	3

FIGURE 11. **B_4, an array that is not the adjacency pattern of any map.**

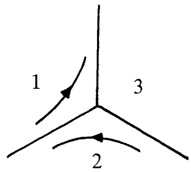


FIGURE 12

just before he encounters country 1, which contradicts the information in the second row of B_4. This observation generalizes to the following **consistency constraint** to which all arrays that describe complete m-maps must conform: *If in the ith row of the array, j is followed by k, then in the kth row, j must be preceded by i.* In other words,

if the ith row is

$$i) \quad \dots \quad j \; k \quad \dots$$

then the kth row is

$$k) \quad \dots \quad i \; j \quad \dots .$$

Surprisingly, there are no other constraints. *Any array that satisfies both the shape and consistency constraints describes a complete m-map.* Moreover, as Heffter demonstrated, this complete m-map verifies the Heawood Conjecture for the surface S_g for $g = (m-3)(m-4)/12$. Since the genus g (number of tunnels) of a surface is necessarily an integer, it follows that the product $(m-3)(m-4)$ must be divisible by 12, from which it follows that m itself, when divided by 12, must yield 0, 3, 4, or 7 as remainder. Since, in general, division by 12 yields one of 12 possible remainders, this approach can yield the appropriate complete m-map for at most one third of the possible values of m. As it happens, even this estimate is overly optimistic. Heffter was aware that an array possessing the additional cyclic structure displayed by A_7 could only be constructed when the remainder of m divided by 12 was 7. All of these limitations notwithstanding, he felt that this was an approach well worth pursuing. He translated the problem into a purely number theoretic one and then showed that the required cyclic arrays A_m exist for those values of m that satisfy conditions i)–iii) below:

i) m leaves remainder 7 when divided by 12,

ii) $(m+2)/3$ is a prime number,

iii) $2^k - 1$ is not divisible by $(m+2)/3$ for $k = 1,2,3,\dots,(m-7)/6$.

Among the values that satisfy these conditions we find the numbers $m = 19, 31, 55, 67, 139, 175, 199$. Consequently, Heffter verified the Heawood Conjecture for the surfaces S_g where $g = 20, 63, 221, 336, 1530, 2451, 3185$. Actually, he accomplished a little more than that. Since $H_{21} = [\frac{1}{2}(7 + \sqrt{1 + 48 \times 21})] = [19.38\dots] = 19 = H_{20}$, and since every map that can be drawn on S_{20} can also be drawn on S_{21}, it follows that Heffter's construction of A_{19} also verifies the Heawood Conjecture for S_{21}, as it does for S_{22} as well. Similarly, the existence of the array A_{199} verifies the Heawood Conjecture for all surfaces with at least 3185 but no more than 3217 tunnels. The algebra Heffter used in his work was deep and difficult enough that he could not decide whether the class of surfaces to which his proof applies was finite or infinite. Even today it is still not known whether Heffter's solution applies to infinitely many surfaces.

Heffter's contributions can be summarized as follows. He pointed out the error in Heawood's paper and thereby attracted the attention of the mathematical world to Heawood's beautiful conjecture. He showed how this geometrical problem could be translated first to a combinatorial one of arrays, and then to an algebraic problem in the theory of numbers. He went on to solve this number theoretic problem in some special cases. Finally, a fact that was not mentioned above, he also confirmed the Heawood conjecture for all surfaces of genus at most 7 by constructing the appropriate arrays.

The resolution of Heawood's conjecture

Nothing was contributed toward the resolution of Heawood's conjecture in the sixty years that followed the publication of Heffter's work. That mathematicians were aware of it is attested to by the fact that it is mentioned in Hilbert and Cohn-Vossen's 1932 book *Geometry and the Imagination* [**12**], where it is called the problem of contiguous regions. The lack of progress on this problem during the first half of this century can be attributed to three factors. First comes its inherent difficulty. Next, some mathematicians believed that the problem had indeed been solved by Heawood. Courant and Robbins, in their 1941 book *What is Mathematics*? [**5**, p. 248] wrote: "A remarkable fact connected with the four color problem is that for surfaces more complicated than the plane or the sphere the corresponding theorems have actually been proved." (As late as 1980, I met a topologist who believed the same.) Finally, and perhaps most significantly, to many mathematicians this problem seemed to represent a blind alley. It had been shown towards the end of the 19th century that the combinatorial approach could be very fruitful in analyzing geometrical problems in all dimensions. However, there were no indications that a solution to Heawood's problem would lead anywhere else. Even the Four Color Problem, which many had discounted for the same reason, has corollaries which mathematicians consider to be interesting, although perhaps not "important."

The first breakthrough to follow Heffter's pioneering work was produced by G. A. Dirac [7], a relative of the famed physicist, in 1952. To understand his accomplishment, let us recall the issue. It was known that every map on the surface of genus g could be colored with no more than H_g colors. To show that this number of colors might actually be required, it would suffice to draw a complete H_g-map on this surface. But is the existence of such a complete map on the surface really necessary? Conceivably the surface S_3 might support a map that requires 9 colors even though it might not admit a complete 9-map. That, after all, was the whole point of the Four Color Problem. It was known that no complete 5-map could be drawn in the plane, but that did not preclude the possibility of some other planar map actually requiring 5 colors. Dirac showed that this situation could not arise on the other surfaces. Specifically, he demonstrated that if the surface of genus g supported a map that required H_g colors, then it would also support a complete H_g-map. Actually, he only proved this for the cases $g=3$ and $g \geqslant 5$. His arithmetic got in the way of the cases $g=0,1,2,4$. Had his proof applied to the case $g=0$, he would have produced a proof of the Four Color Conjecture—yet another near miss. However, the other values he missed, $g=1,2,4$, were already covered by Heffter's work.

Next, in what has been called "a tour de force of combinatorial brilliance" ([**22**, p. 317]), Gerhard Ringel [**19**] constructed in 1954 a set of arrays which confirmed the existence of a complete H_g-map on the surface S_g for all those values of H_g that leave a remainder of 5 upon division by 12. At last the Heawood conjecture had been verified for an infinite number of surfaces.

The special significance that the number 12 has for this problem was pointed out earlier in the discussion of Heffter's work. It was noted there that arrays that satisfied both the shape and the consistency constraints could only be obtained when m left remainders of 0, 3, 4, or 7 upon division by 12. Since the remainder 5 is not one of these, Ringel had to relax the constraints somewhat. He chose to relax the shape constraint as is evident in the arrays $A_5^{(-1)}$ in FIGURE 13

country	adjacency record			
1)	2	4	3	5
2)	3	4	1	5
3)	1	4	2	5
4)	1	2	3	
5)	3	2	1	

FIGURE 13. **$A_5^{(-1)}$, the adjacency pattern of the map $M_5^{(-1)}$.**

1)	6	4	11	8	13	3	9	17	2	16	15	12	7	5	14	10
4)	9	7	14	11	1	6	12	17	5	16	3	15	10	8	2	13
7)	12	10	2	14	4	9	15	17	8	16	6	3	13	11	5	1
10)	15	13	5	2	7	12	3	17	11	16	9	6	1	14	8	4
13)	3	1	8	5	10	15	6	17	14	16	12	9	4	2	11	7
2)	6	14	7	10	5	9	3	16	1	17	12	15	11	13	4	8
5)	9	2	10	13	8	12	6	16	4	17	15	3	14	1	7	11
8)	12	5	13	1	11	15	9	16	7	17	3	6	2	4	10	14
11)	15	8	1	4	14	3	12	16	10	17	6	9	5	7	13	2
14)	3	11	4	7	2	6	15	16	13	17	9	12	8	10	1	5
3)	16	2	9	1	13	7	6	8	17	10	12	11	14	5	15	4
6)	16	5	12	4	1	10	9	11	17	13	15	14	2	8	3	7
9)	16	8	15	7	4	13	12	14	17	1	3	2	5	11	6	10
12)	16	11	3	10	7	1	15	2	17	4	6	5	8	14	9	13
15)	16	14	6	13	10	4	3	5	17	7	9	8	11	2	12	1
16)	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	
17)	14	13	6	11	10	3	8	7	15	5	4	12	2	1	9	

$A_{17}^{(-1)}$

FIGURE 14. **An array that describes the adjacency pattern of a nearly complete map on S_{15}. This map has seventeen countries of which every two, except 16 and 17, are adjacent to each other.**

and $A_{17}^{(-1)}$ in FIGURE 14. Specifically, in $A_5^{(-1)}$, row 4 does not contain the entry 5 and, vice versa, row 5 does not contain the entry 4. This means that in the associated map $M_5^{(-1)}$ of FIGURE 15 the two countries 4 and 5 are not adjacent to each other. The superscript (-1) in the notation for the array and the map records the fact that both the array and the map lack one adjacency to being complete. Similarly, in the map represented by $A_{17}^{(-1)}$, countries 16 and 17 are not adjacent to each other. Now it so happens that the array $A_{17}^{(-1)}$ actually represents an "almost complete" 17-map on the surface S_{15}. Connect the nonadjacent countries 16 and 17 by boring an additional tunnel through S_{15}. This gives us a complete 17-map on the surface S_{16}. Since $H_{16} = 17$ this array implies the validity of the Heawood Conjecture for the surface of genus 16.

Due to the special role played by the number 12, the problem was now recognized as possessing 12 cases, depending on H_g's remainder when divided by 12. By 1961 Ringel had also resolved the cases of remainder 7, 10, and 3, making use of the same technique as in the case of 5. The year 1963 saw the emergence of some new tools. William Gustin [9] discovered a way of encoding arrays in a form that greatly resembles electrical networks. In these, the numbers originally used to label countries are interpreted as denoting the intensity of a (fictitious) current flowing along the branches of the network in the direction indicated by the arrowheads. Kirchhoff's Current Law, which states that the total current entering a node equals the total

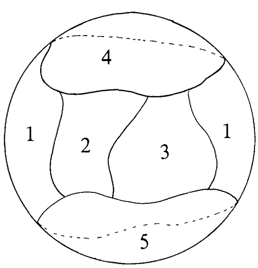


FIGURE 15. $M_5^{(-1)}$, **an almost complete 5-map on the sphere. Every two countries, except 4 and 5 touch each other.**

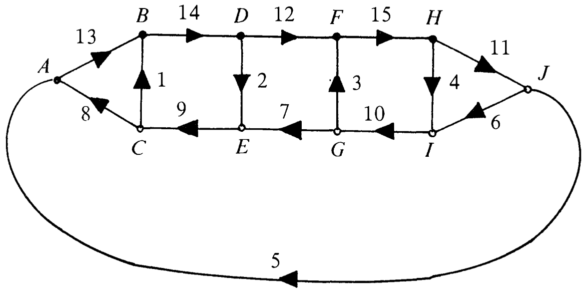


FIGURE 16. CG_{31}, a current graph which encodes a complete 31-map on S_{63}.

current leaving it, is also satisfied by these networks. Because of this resemblance, Gustin's networks have been dubbed "Current Graphs." To indicate the manner in which an array may be stored in a current graph, we decode CG_{31} (FIGURE 16). Choosing our departure point arbitrarily, we start out with the network branch labelled 5 and proceed to node A. From here we could choose to go to either B or C. The correct choice is indicated by the way in which the node is drawn. A solid dot indicates that at this node the traveler should choose the left fork, whereas a hollow dot dictates a choice of the right fork. Hence we go on to B and record a 13 in our logbook, to follow the previous entry 5. Node B is also a solid dot, and so we go to D from here and record a 14 in the logbook. By the time node J is reached, the logbook's entries will be 5,13,14,12,15,11. Since the node J is represented by a hollow dot, we choose the right fork and go on to I, adding the entry 6 to the log. From I we again bear right, toward H, but this time, since we are progressing *against* the arrowhead, instead of recording 4 in the log, we enter $31-4=27$. From the solid dot of H, we choose the left fork to F and record $31-15=16$ in the log. The solid and hollow nodes are so placed that the initial current 5 will not be encountered again before all the possible currents from 1 to 30 have been recorded in the logbook. In other words, this procedure is set up so that each of the branches of the network will be traversed exactly twice, once in each direction. If we consider the above tour as terminating when 5 is reencountered, then the final log is:

$$5,13,14,12,15,11,6,27,16,28,7,29,17,30,8,$$
$$26,20,4,10,3,19,2,9,1,18,23,22,24,21,25.$$

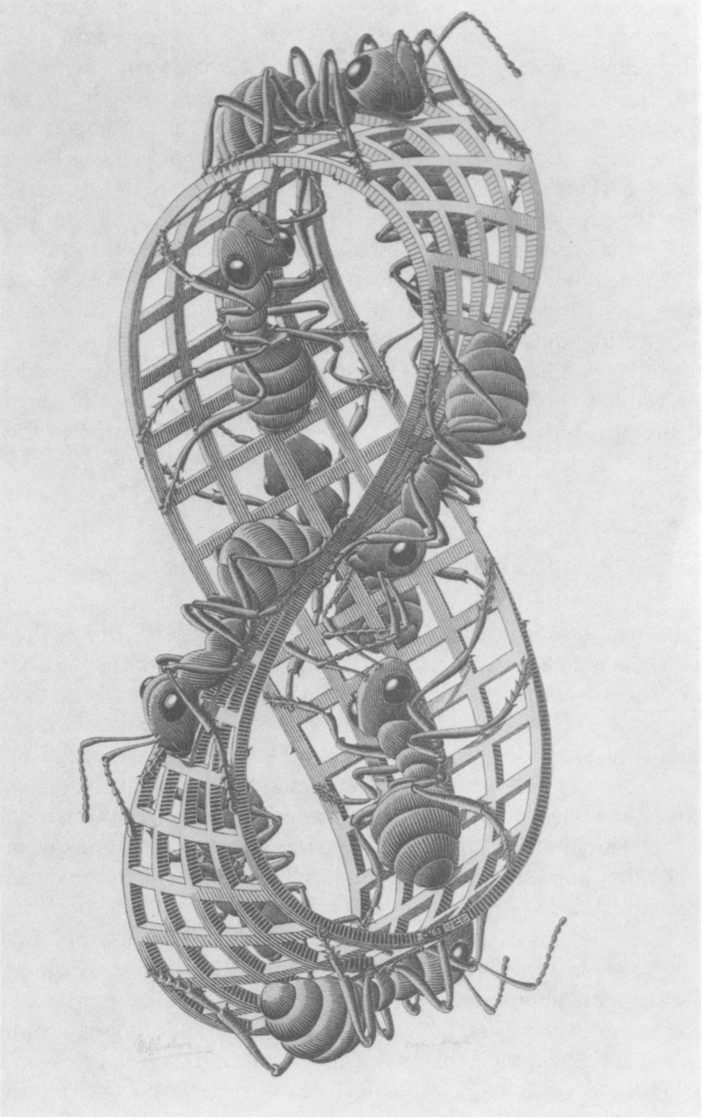
The ith row of the array A_{31} is now obtained by adding i to each entry of the above log, with the stipulation that $31+i$ is to be replaed by i. This array satisfies both the shape and consistency constraints and so it attests to the existence of a complete 31-map on the surface S_{63}. Since $H_{63}=[\frac{1}{2}(7+\sqrt{1+48\times 63})]=31$, the Heawood Conjecture for that surface is verified.

Mysterious as the above procedure may seem, we know why it works. The distribution of the currents among the links of the network and the relative placements of the solid dots versus the hollow dots among the nodes are such as to guarantee that the shape constraint is fulfilled. It can also be shown that Kirchhoff's Current Law is transformed in the array into the consistency constraint. On the other hand, the heuristics underlying the idea that a map can be encoded as an electrical network are still not well understood. It should be pointed out that complex function theory, within which Riemann's surfaces were first recognized, has very strong connections to potential theory. Indeed, some very deep mathematical theorems become "obvious" when translated into statements about electrons and electrical fields. Be that as it may, these current graphs are of course much more tractable than the arrays they represent, and these arrays are in turn much more tractable than the actual maps they represent. Nevertheless, even these graphs are far from easy to find, especially as for some of the cases they have to be slightly modified in order to work. It was not until 1968 that the combined efforts of W. Gustin, R. K. Guy, C. M. Terry, L. R. Welch, and, most of all, G. Ringel and J. W. T. Youngs (see **[20]** for details) resolved all of the remaining eight cases of remainders 0,1,2,4,6,8,9,11. Their work left the cases $H_g=18,20,$

and 23 unresolved, but these gaps were filled within one year by J. Mayer [**16**] (professor of French literature at the University of Montpellier). It took three quarters of a century to verify completely the statement that Heawood felt was too obvious to require justification.

Coloring maps on other surfaces

The work described above completely resolved the coloring problem on Riemann's surfaces, except for the sphere. An issue that has so far been sidestepped is the question of whether there are any other surfaces for which interesting coloring problems could be posed. The reader is reminded that maps of higher complexity were made possible by boring tunnels through the sphere, thus motivating the definition of the Riemann surfaces. Is there anything else that could be done to a sphere to allow for more complex maps? The answer is yes.



Mobius Strip II, woodcut by M. C. Escher, 1963.
Copyright M. C. Escher heirs c/o Cordon Art B. V.-Baarn, Holland.

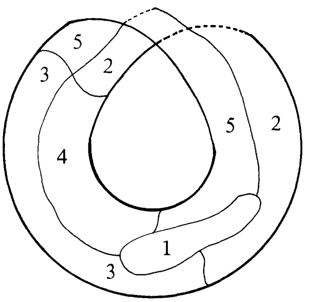


FIGURE 17. $\tilde{M}_5$, **a complete 5-map on the Möbius strip.**

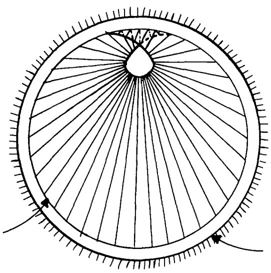


FIGURE 18. **Trying to patch a hole with a Möbius strip.**

The map $\tilde{M}_5$ (FIGURE 17) displays a complete 5-map on the Möbius strip. This strip is obtained by making a 180° twist in a long ribbon and then gluing its two ends to each other. Now this map can be transferred to the sphere in the following manner. Observe that the edge of the Möbius strip consists of a *single* closed loop, as opposed to, say, the two loops that would have formed the edge had the ribbon not been twisted before its ends were glued. If a disk is cut out of the surface of the sphere, a closed loop is left as the edge of the perforated sphere. Since all closed loops are essentially identical, it should be possible to patch the hole in the sphere by sewing its edge to the edge of the Möbius strip. Most of the readers who try to carry out this patching process will soon discover that the twist in the strip has a nasty tendency to get in the way (see FIGURE 18). However, my four-dimensional readers should experience no such difficulties. That extra dimension will provide them with all the room they need to maneuver. For this reason mathematicians have accepted the resulting hybrid as a genuine surface, even though its internal structure prevents it from being realized in our three-dimensional space. It meets the criteria that mathematicians have set for surfaces, namely, they can have no edges and at each point they must be "reasonably" flat.

Of course, once this twisted surface is accepted, one must be prepared to allow for the possibility of adding more than one twist, just as we allowed for the possibility of boring several tunnels in the sphere. The surface obtained by adding g twists to the surface of the sphere is called the **one-sided surface of genus** g, and is denoted by $\tilde{S}_g$. It is one-sided because the twisted patches would allow an ant living on the outside of the sphere to move to its inside simply by walking along a twist in the manner depicted by Escher's famous woodcut. The one-sided surface $\tilde{S}_1$ is the projective plane and $\tilde{S}_2$ is the well-known Klein bottle. By way of contrast, Riemann's surfaces are all two-sided. An ant on the outside has no way of getting into the inside. And what would happen if twists were added to the other two-sided surfaces? Would we then obtain a whole new bewildering collection of surfaces with mixed numbers of tunnels and twists? Fortunately the answer is no. It has been known since the turn of the century that when a twist is added to the two-sided surface of genus g, one simply obtains the one-sided surface of genus $2g+1$. Similarly, when a tunnel is bored into the one-sided surface of genus g, the result is the one-sided surface of genus $g+2$. Moreover, there are no other surfaces. The two sided and the one-sided surfaces comprise the totality of all surfaces (see, for example, [**15**]).

One-sided surfaces were still very new when Heawood formulated his problem. At the time they were probably regarded as a curiosity rather than a significant phenomenon, which may explain why Heawood, as well as others who should have known better, ignored them. They were mentioned by Heffter, but he felt, for good reasons, that once the coloring problem was resolved for the two-sided surfaces, a certain theoretical connection between them and their one-sided siblings could be utilized to solve the same problem for the latter as well. In the event, Heffter's good reasons notwithstanding, history did not bear him out.

The Euler-Poincaré formula for the one-sided surfaces states that

$$n - b + m = 2 - g$$

holds for maps on $\tilde{S}_g$. From this it follows that every map on this surface can be colored with

$$\tilde{H}_g = \left[\tfrac{1}{2}\left(7 + \sqrt{1 + 24g}\right)\right]$$

colors, and it is of course natural to conjecture that every $\tilde{S}_g$ supports a map that actually requires that many colors. This was indeed done in 1910 by the topologist H. Tietze [**21**], who also pointed out that the complete map $\tilde{M}_6$ of FIGURE 19 was implicit in some work by Möbius [**17**]. Since $\tilde{H}_1 = 6$, this map solves the coloring problem for $\tilde{S}_1$. Every map on $\tilde{S}_1$ can be colored with 6 colors, and some such map in fact requires as many as 6 colors.

In 1934, P. Franklin [**8**] uncovered a surprising fact. He showed that while $\tilde{H}_2 = 7$, every map on the Klein bottle $\tilde{S}_2$ could be colored with no more than 6 colors! This was the first known failure of the conjecture that each of the numbers H_g and $\tilde{H}_g$ is indeed required by some map on the corresponding surface. Once such an exception is found, of course, it becomes reasonable to expect others to occur. By 1943, the work of I. N. Kagno [**13**], H. S. M. Coxeter [**6**], and R. C. Bose [**3**] showed that no such failures could occur on $\tilde{S}_3$, $\tilde{S}_4$, $\tilde{S}_5$, $\tilde{S}_6$, or $\tilde{S}_7$. They did this by constructing arrays for the appropriate complete m-maps. These arrays are very similar to the ones used for the description of maps on the two-sided surfaces. The main difference is that the consistency rule is replaced by:

If the ith row is

$$i) \quad \ldots \quad j \quad k \quad \ldots$$

then the kth row is

either $\quad k) \quad \ldots \quad i \quad j \quad \ldots$

or $\quad k) \quad \ldots \quad j \quad i \quad \ldots$

and the jth row is

either $\quad j) \quad \ldots \quad k \quad i \quad \ldots$

or $\quad j) \quad \ldots \quad i \quad k \quad \ldots.$

In 1954, in the same paper that contained his first major contribution towards the resolution of the Heawood Conjecture, Ringel completely solved Tietze's coloring problem for one-sided surfaces. He accomplished this by producing Heffter-style arrays for the requisite complete maps on all of these surfaces. This was a truly formidable achievement. His proof, by the way, showed that the Klein bottle provided the only exception to the rule that, in general, the number $\tilde{H}_g$ is the solution of the coloring problem on $\tilde{S}_g$. The reason for the earlier resolution of the problem for one-sided surfaces is that the consistency constraint for these is considerably less restrictive than the one for the two-sided surfaces. This pattern recurs frequently in the theoretical study of

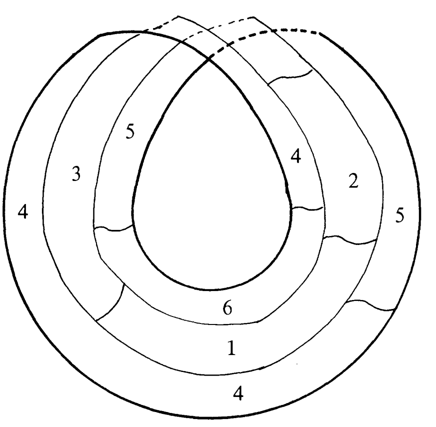


FIGURE 19. **$\tilde{M}_6$, a complete 6-map on the Möbius strip.**

surfaces. Many questions are formulated first in the context of two-sided surfaces, because they are so easily visualized. They are first resolved, however, for one-sided surfaces, despite the fact that the natural habitat of these surfaces is in four-dimensional space.

In 1967 Youngs simplified Ringel's solution by replacing his arrays with current graphs. The complete solution to the problems posed by Heawood and Tietze is now known as the **Ringel-Youngs Theorem**. In 1974 Ringel published his book *Map Color Theorem* [**20**], which contains both the complete details of the solution and an explanation of the underlying theory of current graphs.

I had several reasons in mind when I decided to recount the history of the Ringel-Youngs Theorem. It is one of my favorite theorems, both because it deals with surfaces and because its proof is so rich. Moreover, I felt that its history makes for a good story. Finally, I saw this as a way to bring the reader closer to the way mathematicians actually operate. Problems, be they solved or unsolved, give rise to more problems. We saw the Four Color Problem motivate the Heawood conjecture, and the latter eventually gave rise to Tietze's one-sided analog. The path leading to a problem's solution is often littered with mistakes, such as those made by Kempe and Heawood. It is the good mathematician who can extract useful information from his own and other people's mistakes, and use them as a basis for new investigations. We saw mere notational conventions transformed into crucial breakthroughs, while other promising technical approaches dwindled into blind alleys. The final solution came as a result of fusion of disparate mathematical disciplines and the cooperative efforts of several mathematicians. These are only some of the elements that go into the production of a mathematical proof, but they are also ones that do not appear in the final product.

The author thanks all those readers of earlier drafts of this article for their helpful critical comments.

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NOTES

Three Observations on a Theme: Editorial Note

DAVID A. SMITH
Associate Editor

The history of mathematics is replete with nearly simultaneous, or at least independent, discoveries and rediscoveries, from "little" results amounting to mere observations to major theorems and theories. The two notes following offer, in addition to the insights each contains, an example of Editor Schattschneider's observation that "if you receive one manuscript on a topic, you are likely to receive another one on the same topic in two weeks." As one of the cited "codiscoverers," I would like to provide a backdrop for these two notes. The result discussed here is a "little" one, namely, the resolution of the question: "*If a sufficiently smooth function of two variables has exactly one critical point, and it occurs at a local extremum, is that also a global extremum*?" My interest in the question arose in the following way.

During the period 1976–1981 I had the good fortune to serve as Calculus Project Adviser for the D. C. Heath Co. for the specific purpose of assisting with the development of Philip Gillett's *Calculus and Analytic Geometry*. All of us who teach calculus (authors of books possibly excluded) know that we have a better textbook in our heads than any of the ones we have taught from, and this was my opportunity to get some of my ideas into print without devoting the enormous time and effort required for writing a good book. (I say "some" because, whenever Phil and I disagreed on something, it was *his* book to write.) At some point in our discussions on multivariable calculus, Phil asked me the question posed above. I'm sure we weren't the first to consider it, but we couldn't recall having seen the answer in any of the calculus books we had read. And since one of the principles we hoped to stress was the continuity of the subject, that is, the natural flow of multivariable concepts from the extensive treatment of functions of one variable, it seemed to be a natural question to resolve.

My initial reaction (not very different from that reported by the authors of the following notes) was that the answer to the question (as posed above) was positive, and that I could prove it with a little effort. It didn't take me long to start doubting that first intuitive flash, and I began to visualize (in three dimensions, not as a contour map) something like FIGURE 1 in the Rosenholtz-Smylie note: a hill separated from an infinitely rising wall by a gently sloping valley that never levels off. I saw the cross sections perpendicular to the valley as cubics, the cross section of the hill parallel to the valley as bell shaped, and the valley itself as exponential. I placed the peak of the hill at the origin and the valley along $x=1$, and all that remained was working out appropriate coefficients for the cubics:

$$f(x,y)=e^{-y^2}(2x^3-3x^2+1)+e^{-y}(2x^3-3x^2).$$

Phil was delighted to have an explicit answer to the question and included the result as an exercise in the book (see the Ash article for the reference to the current edition). The student is asked only to confirm the result, not to discover the correct answer. This book has been in print for four years, but an exercise deep inside a standard calculus book is not exactly a prominent form of publication, so it is no surprise that others have asked and answered the same question in their own ways. Each of the notes following presents a somewhat different counterexample to the intuitive "theorem," each adds some insight, and each is illustrated with computer graphics that should straighten out everyone's intuition on this point forever.

A Surface With One Local Minimum

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Consider the following statement.

STATEMENT F. *A smooth surface* $(x, y, f(x, y))$ *with one critical point which is a local, but not a global, minimum must have a second critical point.*

Here smooth may be taken to mean that f is infinitely differentiable. A point (a, b) is a critical point if

$$\frac{\partial f}{\partial x}(a, b) = \frac{\partial f}{\partial y}(a, b) = 0.$$

When asked what we thought of this statement, our first reaction was that it ought to be true since a one-dimensional analogue is. After some thought we came up with an "almost rigorous" argument that supported statement F. Somewhat later, assisted by a geometric idea of P. Ash of St. Joseph's University (the first author's brother), we found a counterexample to statement F. Finally, we applied an important principal of mathematical research which A. Zygmund of the University of Chicago has frequently expounded in his seminar: Never be stopped by a counterexample; instead find out what is really happening.

Guided by this maxim, we were able to add a small hypothesis (suggested by William Browder of Princeton University) which did force the conclusion of statement F to follow. More explicitly, we have

THEOREM T. *Let* f: $R^2 \to R$ *be continuously differentiable and have a local, nonglobal minimum. If, further,* f *is proper* ($f^{-1}(K)$ *is compact whenever* K *is a compact subset of the range) then* f *must have at least one additional critical point.*

The theorem appears to be unpublished folklore, and we will supply a proof after first presenting some thoughts about statement F, then a counterexample.

In considering statement F, an obvious question to ask is: What happens in one dimension? Let f: $R \to R$ be smooth and have a local minimum at 0, for example, $f(0) = 0$ and $f(x) > 0$ for all $|x| < \delta$. If 0 is not a global minimum then we must have $f(a) < 0$ for some a, say $a > 0$. But then $f(\delta/2) > 0$, $f(a) < 0$ and the intermediate value theorem gives $f(b) = 0$ for some b in $(\delta/2, a)$. But $f(0) = f(b) = 0$ so Rolle's Theorem implies the existence of $c \in (0, b)$ with $f'(c) = 0$. Thus f has a second critical point at c.

Encouraged by this evidence for statement F, let us perform a thought experiment. Suppose we pour water onto the surface from a spout located directly above the critical point. The water will steadily rise. Evidently it must overflow sooner or later if the local minimum is not absolute. The point at which the overflow first occurs must be a second critical point. This argument has strong intuitive appeal, but the idea that the "bowl" around the local minimum has finite volume is implicitly incorporated into the assumption of eventual overflow.

The following counterexample shows that statement F is false; although discovered independently, it is somewhat similar to one given by David Smith [**1**, p. 750]. Define

$$f(x, y) = \frac{-1}{1 + x^2} + (2y^2 - y^4)\left(e^x + \frac{1}{1 + x^2}\right). \tag{1}$$

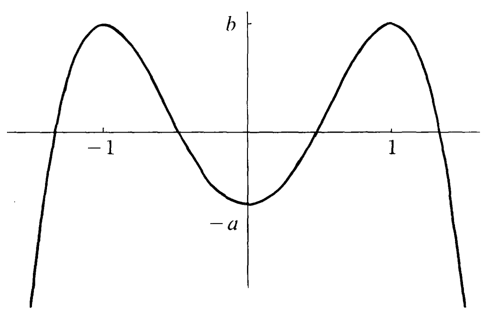

FIGURE 1. **Section $x=$ constant.**

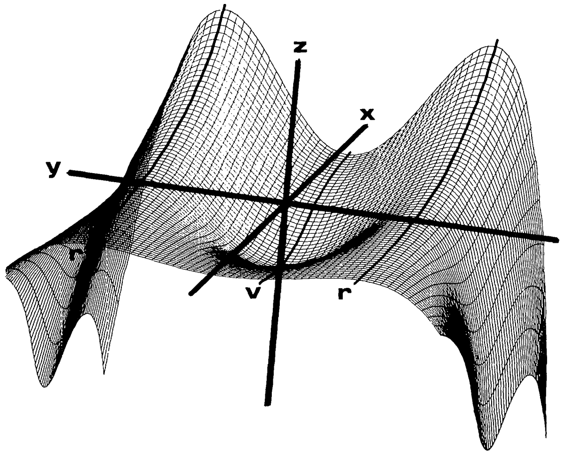

FIGURE 2

The function f is as differentiable as you like (in fact, it is real-analytic). The point $(0,0)$ is a local, not global minimum, and there are no other critical points.

To find any critical point(s) of f, first consider sections of the form $x=\text{constant}$. Then $f=f(y)=-a+(2y^2-y^4)(b+a)$ has critical points at $y=0,1$, and -1. Now consider sections of the form $y=\text{constant}$. As FIGURE 1 shows, we need only look at the sections $y=1$, $y=0$, and $y=-1$, since nowhere else is $\partial f/\partial y=0$. On the sections $y=-1$, $y=1$, $f=e^x$ so $\partial f/\partial x=e^x$ is always positive. On the section $y=0$, $f(x)=-1/(1+x^2)$, so $\partial f/\partial x=2x/(1+x^2)^2=0$ only at $x=0$. Thus $(0,0)$ is the only critical point.

Since $f(0,0)=-1>-17=f(0,2)$, the point $(0,0)$ is not a global minimum. It only remains to show that $(0,0)$ is indeed a local minimum. We have

$$f(x,y)=\left[\frac{-1}{1+x^2}\right]+(2y^2-y^4)\left(e^x+\frac{1}{1+x^2}\right)$$

$$=\left[-1+\frac{x^2}{1+x^2}\right]+y^2(2-y^2)\left(e^x+\frac{1}{1+x^2}\right)$$

$$=f(0,0)+\left\{\frac{x^2}{1+x^2}+y^2(2-y^2)\left(e^x+\frac{1}{1+x^2}\right)\right\}. \tag{2}$$

If (x,y) is in the unit disc about $(0,0)$, then the quantity in curly brackets in (2) is positive except when $x=y=0$. This shows $(0,0)$ to be a local minimum.

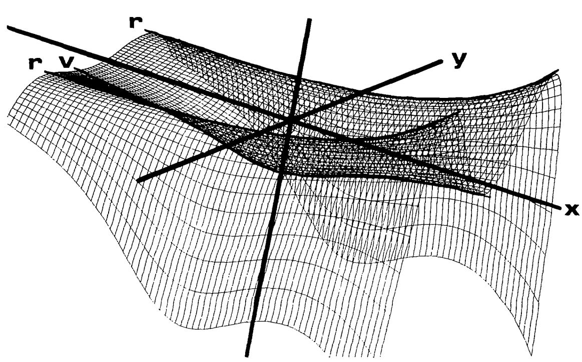

FIGURE 3

We restate the above proof geometrically, using FIGURES 2 and 3. (The 3 coordinate axes are separately scaled here to bring out the salient features.) Each section "parallel" to the y-axis has the shape shown in FIGURE 1 with one dimensional critical points occurring on the ridges labeled r and on the valley bottom labeled v. Since the ridges grow like e^x the tangent plane can be horizontal only at some point of v. The only such point of v is the place where the z axis pierces v, which is the local minimum.

Proof of Theorem T

Assume f satisfies the hypotheses of the theorem. Let f have its local minimum at $A \in R^2$ and let $B \in R^2$ be such that $f(B) < f(A)$. Pour water onto the surface determined by f, above the point A. Then either (i) the water level will rise to arbitrarily great heights, or (ii) the water level will asymptotically approach a finite height h (as actually happens for f defined by (1)), or (iii) the water will overflow. We will proceed to eliminate cases (i) and (ii), in which case our earlier argument will become the proof of the theorem.

In R^2 let $\overline{AB}$ be the line segment joining A and B. Then the continuous function f attains a maximum, say m, on the compact set $\overline{AB}$. The water cannot rise to a height greater than m without spilling so that case (i) is impossible.

If case (ii) were to occur, then the water would be held by a reservoir which would lie over a subset S of the compact set $f^{-1}([f(A), h])$ and whose depth would be everywhere less than $h - f(A)$. Such a reservoir would have finite volume (less than the product of the measure of S with $h - f(A)$). Since we may pour as much water as we like, case (ii) is impossible.

References

[1] Philip Gillett, Calculus and Analytic Geometry, 2nd ed., D. C. Heath, Lexington, Mass., 1984.

"The Only Critical Point in Town" Test

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When searching for absolute extrema of functions of a single variable, it is often convenient to apply the well-known "Only Critical Point in Town" Test: *If f is a continuous function on an interval, which has a local extremum at x_0, and x_0 is the only critical point of f, then f attains an absolute extremum at x_0.* A natural question which arises is "Is the corresponding statement true for functions of two variables (say defined over the entire plane)?" Since our colleagues were evenly split on the question (both halves being quite adamant), and neither a proof nor a counterexample was readily available, we set to work trying to find one.

Progress came slowly at first. Then one bright Monday morning we exchanged pictures of what we thought the level curves of a counterexample might look like. And believe it or not, we both had the same picture!—right down to the location of the mountain, the river bed, and the cliff! It looked something like FIGURE 1. Most of our colleagues were convinced by our picture, but a few remained rightfully skeptical: They wanted a formula. We did too.

Then we noticed something—our level curve picture seemed to have a "saddle point at infinity." So we tried to mimic this.

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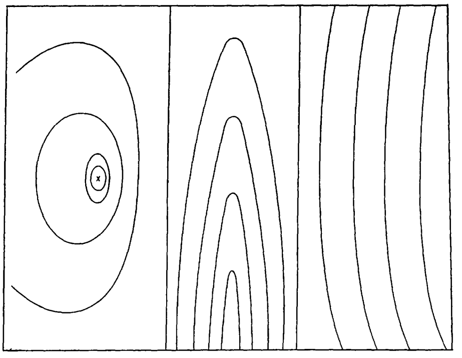

FIGURE 1

Starting with the surface

$$g(x,y) = 3xy - x^3 - y^3,$$

which has a local maximum at (1,1) and a saddle point at (0,0), to "push the saddle point to infinity," we simply took the function

$$f(x,y) = g(x,e^y) = 3xe^y - x^3 - e^{3y}.$$

With this function it is easily seen that the point (1,0) is the only critical point in the plane and that f attains a local maximum there. But it is clearly not an absolute maximum since $f(x,0) \to \infty$ as $x \to -\infty$.

In FIGURE 2 you may look at the "entire" graph of f which we have "scrunched" for easy viewing.

We would like to thank Hans Reddinger and some of his University of Wyoming "toys" for the great graphics.

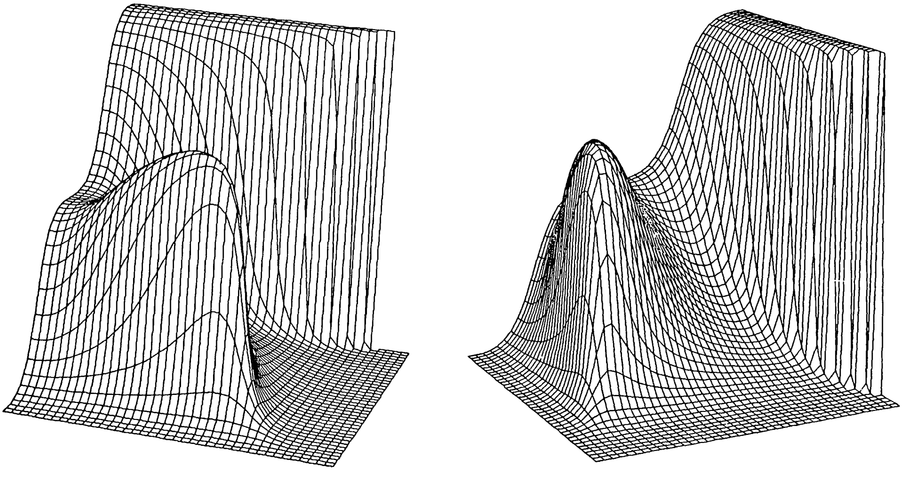

FIGURE 2. **Views of the "scrunched" version of $z = 3xe^y - x^3 - e^{3y}$, that is, $z = \arctan[3e^{\tan y}\tan x - \tan^3 x - e^{3\tan y}]$ for $-\pi/2 < x,\ y < \pi/2$.**

Averages on the Move

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This note is concerned with how averages change when the items being averaged are transformed in a simple way. To make this more explicit we remind the reader of the definitions of four basic averages or means. Then we turn to a specific example, using the Celsius and Fahrenheit temperature scales to illustrate the problems involved.

Denote the **arithmetic mean, geometric mean, harmonic mean**, and the **root mean square** by A, G, H, and R, respectively. For a set $\{a_1, a_2, \dots, a_n\}$ of positive real numbers these four means may be defined by the equations

$$\begin{aligned} A(a_1, a_2, \dots, a_n) &= \frac{1}{n}(a_1 + a_2 + \cdots + a_n) \\ G(a_1, a_2, \dots, a_n) &= (a_1 a_2 \cdots a_n)^{1/n} \\ H(a_1, a_2, \dots, a_n) &= \left(\frac{1}{n}\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right)\right)^{-1} \\ R(a_1, a_2, \dots, a_n) &= \sqrt{\frac{1}{n}(a_1^2 + a_2^2 + \cdots + a_n^2)}\ . \end{aligned} \tag{1}$$

To avoid trivial exceptions to our results, we assume throughout that $a_1, a_2, \dots, a_n$ are not all equal.

For example, with $\{a_1, a_2, \dots, a_6\} = \{1, 2, 3, 12, 18, 36\}$, these means are $A = 12$, $G = 6$, $H = 3$, and $R = \sqrt{889/3}$. These values illustrate the well-known inequalities [**3**, pp. 36, 37] that $H < G < A < R$.

Consider now the sequence of temperatures $t_1, t_2, \dots, t_{720}$ measured in degrees Celsius at intervals of two minutes throughout an entire day, with the first reading at 12:01 a.m. and the last at 11:59 p.m. Let these same temperatures, measured in Fahrenheit degrees, be denoted by $T_1, T_2, \dots, T_{720}$, so that the well-known relationship

$$T_j = 32 + \frac{9}{5} t_j, \qquad j = 1, 2, \dots, 720 \tag{2}$$

holds. Assuming all temperatures positive, let the four means defined by (1) for the Celsius data t_j be denoted by A_C, G_C, H_C, and R_C and for the Fahrenheit data by A_F, G_F, H_F, and R_F. One might expect these averages to also satisfy relation (2); that is, the equations

$$A_F = 32 + \frac{9}{5} A_C, \qquad G_F = 32 + \frac{9}{5} G_C, \qquad H_F = 32 + \frac{9}{5} H_C, \qquad R_F = 32 + \frac{9}{5} R_C \tag{3}$$

might seem reasonable. Our insight into the real meaning of equations (1) and (2) can be tested by deciding which, if any, of equations (3) are valid, without turning to numerical examples, of course. If the decision is negative in any case, then we should decide whether the equality sign should be replaced by a greater than sign or a less than sign.

To simplify these questions, let us separate the linear relationship (2) into two parts: first, multiplication by a constant, then addition of a constant. It is not difficult to see directly from definitions (1) that under the first operation, the averages behave quite regularly. That is, for any positive constant k we have the homogeneity property:

$$F(ka_1, ka_2, \ldots, ka_n) = kF(a_1, a_2, \ldots, a_n) \tag{4}$$

when F is any of the four means A, G, H, or R.

On the other hand, what is the outcome if the positive numbers $a_1, a_2, \ldots, a_n$ are increased by a positive constant, say c? Does the equation

$$F(a_1 + c, a_2 + c, \ldots, a_n + c) = c + F(a_1, a_2, \ldots, a_n) \tag{5}$$

hold with A or G or H or R in place of F? Referring to our earlier example, with the given set of numbers $\{1,2,3,12,18,36\}$, if we increase each number in the set by 10 to get $\{11,12,13,22,28,46\}$, do the means increase by 10, or something less or something more?

The answers to the questions we have raised are summarized in the Theorem below. First, equation (5) holds with F replaced by the arithmetic mean A, as is readily verified from the definition. However, in the other three cases we have inequalities.

THEOREM. *If A, G, H and R are the means defined by equations* (1), *then for any positive constant c,*

$$A(a_1 + c, a_2 + c, \ldots, a_n + c) = c + A(a_1, a_2, \ldots, a_n), \tag{6}$$

$$G(a_1 + c, a_2 + c, \ldots, a_n + c) > c + G(a_1, a_2, \ldots, a_n), \tag{7}$$

$$H(a_1 + c, a_2 + c, \ldots, a_n + c) > c + H(a_1, a_2, \ldots, a_n), \tag{8}$$

$$R(a_1 + c, a_2 + c, \ldots, a_n + c) < c + R(a_1, a_2, \ldots, a_n). \tag{9}$$

Before proving the Theorem, we note some of its implications. In our example of Celsius and Fahrenheit temperatures, we conclude that in (3), the first equation holds, but the equality sign should be replaced by " $>$ " in the second and third cases, and by " $<$ " in the fourth. As another example, consider the situation where all the employees in a plant are given a raise. If the raise is the same dollar amount for each worker, as in a flat cost-of-living adjustment, we are moving from incomes $a_1, a_2, \ldots, a_n$ to $a_1 + c, a_2 + c, \ldots, a_n + c$. On the other hand, if all the employees are given the same percentage increase, we move from $a_1, a_2, \ldots, a_n$ to $ka_1, ka_2, \ldots, ka_n$. In the latter case, equation (4) tells us that the mean of the new salaries is just k times the mean of the old salaries, no matter which mean is used for the computation. However, for the case of a constant increase c in salary, the Theorem states that the 'mean' of the new salary equals the mean of the old salary plus c only in the case of the mean A.

To prove the Theorem, it turns out to be more convenient to prove slightly stronger results, which we formulate in terms of derivatives. For the arithmetic mean we have the rather evident equation

$$A(a_1 + x, a_2 + x, \ldots, a_n + x) = x + A(a_1, a_2, \ldots, a_n).$$

Differentiating with respect to the variable x we have

$$\frac{dA}{dx} = \frac{d}{dx} A(a_1 + x, a_2 + x, \ldots, a_n + x) = 1. \tag{10}$$

In contrast with this, similar consideration of derivatives of the other means under discussion shows they satisfy inequalities.

LEMMA. *Let* $G(x) = G(a_1 + x, \ldots, a_n + x)$, $H(x) = H(a_1 + x, \ldots, a_n + x)$, *and* $R(x) = R(a_1 + x, \ldots, a_n + x)$. *Each of these is a differentiable function of x, and*

$$\frac{dG}{dx} > 1, \tag{11}$$

$$\frac{dH}{dx} > 1, \tag{12}$$

and

$$\frac{dR}{dx} < 1. \tag{13}$$

Each of the inequalities (11), (12), (13) is proved separately, and we postpone their proof to the next section. Using the Lemma, it is easy to establish inequalities (7), (8) and (9). First, apply the mean value theorem to the differentiable function $G(x)$ on the interval $[0, c]$. This says

$$\frac{G(a_1 + c, a_2 + c, \ldots, a_n + c) - G(a_1, a_2, \ldots, a_n)}{c - 0} = G'(a_1 + \theta, a_2 + \theta, \ldots, a_n + \theta)$$

for some value of θ between 0 and c. But since (by (11)) $G' > 1$, simple algebra yields inequality (7). The same argument can be applied to the functions $H(x)$ and $R(x)$, using inequalities (12) and (13) to prove (8) and (9), respectively. Note that the relations in the Lemma are stronger than (7), (8) and (9), which imply only that the derivatives satisfy $G' \geqq 1, H' \geqq 1, R' \leqq 1$.

From (10) and the Lemma we have at once that

$$\frac{dG}{dx} > \frac{dA}{dx} > \frac{dR}{dx} \text{ and } \frac{dH}{dx} > \frac{dA}{dx} > \frac{dR}{dx}. \tag{14}$$

These inequalities stand in contrast to the well-known results $H < G < A < R$. At first glance there may appear to be something strange here. On the one hand,

$$G(a_1 + x, a_2 + x, \ldots, a_n + x) - A(a_1 + x, a_2 + x, \ldots, a_n + x) < 0$$

no matter how large x is, and yet the difference $G - A$ is an increasing function of the positive variable x, because of the first inequality in (14). Similarly, $R - A$ is positive no matter how large x is, but it is a decreasing function of x. This suggests that we look at the limits of these functions as x tends to infinity. It turns out that the equation

$$\lim_{x \to \infty} \left[F(a_1 + x, a_2 + x, \ldots, a_n + x) - A(a_1 + x, a_2 + x, \ldots, a_n + x) \right] = 0 \tag{15}$$

holds with any one of A, G, H, or R in place of F. This assertion follows from the fact that

$$\lim_{x \to \infty} \left[F(a_1 + x, a_2 + x, \ldots, a_n + x) - x \right] = A(a_1, a_2, \ldots, a_n) \tag{16}$$

holds with any one of A, G, H, or R in place of F. Although this is not unexpected in the case $F = A$, the authors found the other three cases of (16) rather surprising. These results about limits are established in the last section.

Returning to (14), it might be asked whether there is a single chain of inequalities for the four derivatives, similar to $H < G < A < R$ for the functions. The answer is no, because there is no simple inequality between the derivatives of H and G. For example, it is readily verified that for $x = 1$,

$$\frac{d}{dx} H(3 + x, 8 + x) > \frac{d}{dx} G(3 + x, 8 + x),$$

whereas

$$\frac{d}{dx} H(3 + x, 99 + x) < \frac{d}{dx} G(3 + x, 99 + x).$$

A fuller explanation of this is given in the next section.

Inequalities for the derivatives of G, H and R

In this section we establish (11), (12) and (13) and make further observations about these inequalities and their implications. To prove (11), we first note that the derivative of the product

$$\prod_{i=1}^{n} (a_i + x) \tag{17}$$

is the $(n-1)$st elementary symmetric polynomial of the $a_i + x$, denoted here by S_{n-1}. (For example, in the case $n=3$ the polynomial S_2 would be $S_2=(a_1+x)(a_2+x)+(a_1+x)(a_3+x)+(a_2+x)(a_3+x)$.) In general,

$$S_{n-1}=\sum_{i=1}^{n}\left(\prod_{j\neq i}(a_j+x)\right). \tag{18}$$

Since the product in (17) is G^n, where $G=G(a_1+x,\dots,a_n+x)$, we have

$$\frac{d}{dx}G^n=nG^{n-1}\frac{dG}{dx}=S_{n-1}.$$

To prove that $dG/dx>1$ it suffices to show that $S_{n-1}/n>G^{n-1}$. This is nothing more than the arithmetic mean-geometric mean inequality applied to the n terms of S_{n-1} in (18). The geometric mean of these n terms is the nth root of their product, which is $\{(a_1+x)(a_2+x)\cdots(a_n+x)\}^{n-1}$, or $G^{n(n-1)}$, because each a_j+x appears in exactly $n-1$ of the terms of S_{n-1}.

(The inequality $S_{n-1}/n>G^{n-1}$ which we just proved appears in the classic treatise [**2**], as part of Theorem 52 on page 52.)

A special case of the inequality (7) was pointed out by the referee as problem #305 in the USSR Olympiad Problem Book [**4**, p. 72], which states

> Let $a_1,\dots,a_n$ be positive numbers, and let g be their geometric mean, $g=\sqrt[n]{a_1a_2\cdots a_n}$. Prove that $(1+a_1)(1+a_2)\cdots(1+a_n)\geqq(1+g)^n$.

The symbol g here is the same as $G(a_1,a_2,\dots,a_n)$ in our notation. To prove the inequality in the problem, we raise both sides of (7) to the nth power, with $c=1$:

$$\{G(1+a_1,1+a_2,\dots,1+a_n)\}^n>\{1+G(a_1,a_2,\dots,a_n)\}^n=(1+g)^n. \tag{19}$$

The first term here is just $(1+a_1)(1+a_2)\cdots(1+a_n)$, which solves the Olympiad problem. However, we remark on the slight discrepancy in the inequalities, which is strict in (19), whereas the problem allows for equality. The reason for this is that we are assuming throughout this paper that $a_1,a_2,\dots,a_n$ are not all equal.

Next we prove (12). From the definition of harmonic mean in (1), we observe that $H=H(a_1+x,a_2+x,\dots,a_n+x)$ has the property

$$nH^{-1}=(a_1+x)^{-1}+(a_2+x)^{-1}+\cdots+(a_n+x)^{-1}. \tag{20}$$

Differentiating (20) with respect to x we have

$$\frac{d}{dx}(nH^{-1})=-nH^{-2}\frac{dH}{dx}=-\sum_{j=1}^{n}(a_j+x)^{-2}. \tag{21}$$

To prove that $dH/dx>1$ we establish that the absolute value of the final sum in (21) exceeds nH^{-2}. Writing b_j for $(a_j+x)^{-1}$ we see that this amounts to proving $\Sigma b_j^2>nH^{-2}$. Now (20) implies

$$nH^{-2}=\left\{(a_1+x)^{-1}+(a_2+x)^{-1}+\cdots+(a_n+x)^{-1}\right\}^2/n=\left(\sum_{j=1}^{n}b_j\right)^2/n.$$

Hence it suffices to prove

$$\sum_{j=1}^{n}b_j^2>\left(\sum_{j=1}^{n}b_j\right)^2/n \tag{22}$$

where the b_j are positive numbers, not all equal. This well-known result is a special case of the Cauchy Inequality (see page 73 of Beckenbach and Bellman [**1**], and take $a_1=a_2=\cdots=a_n=1$ in their notation). It is easily proved by squaring out the sum on the right side of (22) and replacing each product $2b_ib_j$ by $b_i^2+b_j^2$, which is larger except in the case $b_i=b_j$.

We noted earlier that the inequality $H'>G'$ holds in one numerical case, but $G'>H'$ in another. The referee has kindly drawn our attention to the following argument which explains the situation more fully. We restrict ourselves to the case $n=2$, but use the notation A, G, H as in the Lemma, so that $A=A(a_1+x, a_2+x)$, for example, with $a_1 \neq a_2$. By simple calculus we find that the derivatives satisfy the equations

$$H' = 2-(G')^{-2} \quad \text{and} \quad G' = A/G.$$

Thus $G'>1$. Writing z and y, respectively, for H' and G' we find that the graph of the equation $z=2-y^{-2}$ for $y>1$ reveals why sometimes one derivative is larger, sometimes the other. The graphs of $z=2-y^{-2}$ and $z=y$ have intersection points at $y=1$ and $y=(1\pm\sqrt{5})/2$, because $y=2-y^{-2}$ amounts to $y^3-2y^2+1=0$ or $(y-1)(y^2-y-1)=0$. The graphs reveal that $z>y$ for $1<y<(1+\sqrt{5})/2$, whereas $z<y$ for $y>(1+\sqrt{5})/2$. Since $y=G'=A/G$, it follows that $H'>G'$ in case $2A<(1+\sqrt{5})/G$, whereas $H'<G'$ in case $2A>(1+\sqrt{5})G$.

We now prove (13). Writing R for $R(a_1+x,\dots,a_n+x)$, we see that $nR^2=\Sigma(a_j+x)^2$. Hence we get

$$\frac{d}{dx}(nR^2) = 2nR\frac{dR}{dx} = 2\sum_{j=1}^{n}(a_j+x).$$

To prove that $dR/dx<1$ we show that $\Sigma(a_j+x)<nR$, that is,

$$\sum_{j=1}^{n}(a_j+x) < n\sqrt{\frac{1}{n}\sum_{j=1}^{n}(a_j+x)^2}\,.$$

Squaring both sides and replacing a_j+x by b_j, we find that this is equivalent to $(\Sigma b_j)^2<n\Sigma b_j^2$, which amounts to (22), a result already proved.

Limits

We now establish the results in (15) and (16). Since

$$x = A(a_1+x, a_2+x,\dots,a_n+x) - A(a_1,a_2,\dots,a_n)$$

it is clear that (16) implies (15). Hence it suffices to prove (16). Furthermore, if we establish (16) in the cases $F=H$ and $F=R$, it follows at once for $F=G$, because G lies between H and R, that is, $H<G<R$.

We shall use a simple result from calculus, namely,

$$\lim_{x\to\infty}\frac{c_m x^m + c_{m-1}x^{m-1}+\cdots+c_0}{k_m x^m + k_{m-1}x^{m-1}+\cdots+k_0} = \frac{c_m}{k_m}, \tag{23}$$

assuming that $k_m \neq 0$. To prove (16) in the case $F=H$, we write $H=H(a_1+x,\dots,a_n+x)$ in the form $H=nG^n/S_{n-1}$ where G^n is the product in (17) and S_{n-1} is given in (18). Thus we can write

$$\lim_{x\to\infty}(H-x) = \lim_{x\to\infty}\frac{nG^n - xS_{n-1}}{S_{n-1}}. \tag{24}$$

Now S_{n-1} is a polynomial in x of degree $n-1$. The coefficient of x^{n-1} is n, and the coefficient of x^{n-2} is $(n-1)(a_1+a_2+\cdots+a_n)$. The coefficient of x^n in G^n is 1, and the coefficient of x^{n-1} is $a_1+a_2+\cdots+a_n$. Hence in $nG_n - xS_{n-1}$ the terms of degree n cancel, and by applying (23) to (24) we get $\lim_{x\to\infty}(H-x)=(a_1+a_2+\cdots+a_n)/n$, and this is $A(a_1,a_2,\dots,a_n)$ as claimed in (16).

To prove (16) in the case $F=R$, we write

$$\lim_{x\to\infty}[R(a_1+x,\dots,a_n+x)-x] = \lim(R-x) = \lim\frac{R^2-x^2}{R+x}. \tag{25}$$

Expanding R^2-x^2 as a quadratic polynomial in x, we notice that the x^2 terms cancel, so that

$$R^2-x^2 = 2(a_1+a_2+\cdots+a_n)x/n + (a_1^2+a_2^2+\cdots+a_n^2)/n.$$

It follows that

$$\frac{R^2 - x^2}{R + x} = \frac{2(a_1 + a_2 + \cdots + a_n)/n + (a_1^2 + a_2^2 + \cdots + a_n^2)/nx}{R/x + 1} \tag{26}$$

where we have divided numerator and denominator by x. As x tends to infinity we note that $\lim R/x = 1$ from the calculation

$$\frac{R}{x} = \sqrt{\frac{1}{n}\left[\left(\frac{a_1}{x} + 1\right)^2 + \left(\frac{a_2}{x} + 1\right)^2 + \cdots + \left(\frac{a_n}{x} + 1\right)^2\right]}.$$

Hence from (25) and (26) we have

$$\lim(R - x) = \lim\frac{R^2 - x^2}{R + x} = \frac{2(a_1 + a_2 + \cdots + a_n)/n + 0}{1 + 1} = A(a_1, a_2, \ldots, a_n),$$

as was claimed in (16).

References

[1] Edwin Beckenbach and Richard Bellman, An Introduction to Inequalities, New Mathematical Library of the MAA, vol. 3, 1961.

[2] G. H. Hardy, J. E. Littlewood, and G. Pólya, Inequalities, Cambridge University Press, 1959.

[3] Ivan Niven, Maxima and Minima without Calculus, Dolciani Mathematical Expositions of the MAA, no. 6, 1981.

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The Fifteen Billiard Balls— a Case Study in Combinatorial Problem Solving

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Balls bearing the numbers from 1 to 15 are on a billiard table. The object of the "game" is to knock all fifteen off the table (into the pockets), where any one of the fifteen balls can be knocked off *first*, but thereafter the next ball to go must be numbered consecutively to one which is already pocketed. Thus if "3" is the first to go, the next one to be removed can be either "2" or "4". If "3" and "4" are off the table, the next one to go can be either "2" or "5". If "3", "4", "2", and "1" are gone, the next one to go *must* be "5", and thereafter "6", then "7", etc., up to "15". (We do not regard "1" and "15" as adjacent!) The question is: *how many different sequences are permitted for removing all fifteen balls from the table*?

The permitted sequences are all permutations of the numbers from 1 to 15, so $15! = 1,307,674,368,000$ is a trivial upper bound on the number of permitted sequences. To get a tighter bound, we observe that while the first ball to go can be any of fifteen, each turn thereafter is a choice of *at most two*; and the very last to go is the only one left, so it is a "choice" of 1. This gives $15 \times 2^{13} \times 1 = 122,880$ as a more realistic upper bound. But how to account for the "sometimes 2, sometimes 1" nature of those intermediate cases?

It follows that

$$\frac{R^2 - x^2}{R + x} = \frac{2(a_1 + a_2 + \cdots + a_n)/n + (a_1^2 + a_2^2 + \cdots + a_n^2)/nx}{R/x + 1} \tag{26}$$

where we have divided numerator and denominator by x. As x tends to infinity we note that $\lim R/x = 1$ from the calculation

$$\frac{R}{x} = \sqrt{\frac{1}{n}\left[\left(\frac{a_1}{x} + 1\right)^2 + \left(\frac{a_2}{x} + 1\right)^2 + \cdots + \left(\frac{a_n}{x} + 1\right)^2\right]}.$$

Hence from (25) and (26) we have

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The brute force solution

Suppose the first ball to go has the number k. If $k=1$ or $k=15$, there is only one way to complete the sequence. If $k=2$, the remaining sequence will be ",3, 4, 5, 6, ..., 14, 15," except that ball number 1 can be inserted at any of the positions in the sequence indicated by a comma, from before 3 to after 15, for a total of 14 *different sequences*. By symmetry, we have the same result if the initial ball has the number $k=14$.

For the general case, if k is the number of the first ball to go off the table, we have the higher numbers, which must be removed in the relative order $k+1, k+2, \ldots, 14, 15$; and the lower numbers, which must be removed in the relative order $k-1, k-2, \ldots, 2, 1$. The number of ways of interspersing these two subsequences is then $\binom{14}{k-1}$, since we must designate which $k-1$ of the fourteen remaining turns (after the first ball has been removed) will involve *lower* numbered balls, and that designation uniquely specifies the rest of that sequence, since there is a unique relative order among the lower-numbered balls, and a unique relative order among the remaining, higher-numbered balls. Hence *the total number of permitted sequences is*:

$$1+14+\binom{14}{2}+\binom{14}{3}+\binom{14}{4}+\cdots+\binom{14}{14}=\sum_{j=0}^{14}\binom{14}{j}=(1+1)^{14}=2^{14}=16{,}384.$$

The simple form of the answer (it would have been 2^{n-1} if we had started with n billiard balls) suggests that there should be a much easier way of arriving at it.

The simple solution

In games (and puzzles) which consist of a finite number of **moves**, where each move involves a "decision" constrained by the rules of the game, it is typically the case that the analysis of the game is simplified by starting with the *last* move (the "winning"—or "losing"—move) and working backward.

In our billiard game, the very last ball to go in must be either "1" or "15", a binary "choice" (if we are running the videotape of this game in reverse!). If the last ball is "1", its predecessor must have been either "2" or "15", again a binary choice. (Similarly, if the last ball was "15", its predecessor was either "1" or "14".) In fact, as we view the game in *reverse*, we see that at each turn there is a binary choice: either the highest or the lowest numbered ball off the table will be the "next" to reappear. And this proceeds all the way back to where we see only one ball (the first ball) still on the table, which will be the unique "choice" for the "final" stage of our reverse process. So the number of possible sequences is trivially 2^{14} (or 2^{n-1}, if we had started with n billiard balls).

A simple "forward" solution

Now that we know our problem not only has a simple answer, but a simple way of arriving at it "in reverse", we can look for a simple "forward" solution. There are 14 **transitions** from one ball to the next as we remove all 15 balls from the table. If the transition is to a higher-numbered ball, let us represent it by $+$; if to a lower-numbered ball, by $-$. There are 2^{14} sequences of $+$'s and $-$'s of length 14, and we can show that they correspond precisely to the allowed sequences of billiard balls. For suppose that there are m minuses, and therefore $14-m$ plusses. Then the first ball to go off the table had to bear the number $k=m+1$, because the minuses in the sequence correspond to transitions to balls numbered lower than k, and the plusses to balls numbered higher than k. Moreover, starting at k, all the sequences of $m=k-1$ minuses and $14-m$ plusses precisely correspond to the distinct ways which are allowed to complete the sequence.

Note that this is really a restatement of the "brute force" (forward) solution, but with a simplified way of counting to arrive at 2^{14}. Reading the 2^{14} sequences of $+$'s and $-$'s *backward*, we have an obvious model for the "simple" (reverse) solution, where "$+$" means "remove a ball at the high end" and "$-$" means "remove a ball at the low end", in progressing from the fifteenth turn back to the second turn.

A more general problem

How many different ways (sequences), $s(t)$, are there to remove the first t of the fifteen balls from the billiard table, subject to our previous rules, for $1 \leqslant t \leqslant 15$? We know that $s(1) = 15$ and $s(15) = 2^{14}$. It turns out that

$$s(t) = (16 - t) \cdot 2^{t-1}, \quad \text{for } 1 \leqslant t \leqslant 15.$$

(If we had started with n balls on the table, and the same basic rules, we would have $s(t) = (n + 1 - t) \cdot 2^{t-1}$, for $1 \leqslant t \leqslant n$.) Proving this formula is one of the harder ways to solve the original problem, but it is certainly doable.

Imagine a snake consisting of 15 segments numbered consecutively from *head* (#1) to *tail* (#15), where the segment numbers correspond to the billiard ball numbers. For the sequences of length t, we visualize the snake as having swallowed the last $t - 1$ segments from its tail end, so that segments $1, 2, 3, \ldots, t - 1$ now coincide with segments $15 - t + 1, 15 - t + 2, 15 - t + 3, \ldots, 15$, respectively. (For $t > 7$, the snake is looped through itself more than once!) From any of the $16 - t$ "distinct" segments, we can exactly represent the billiard sequences of length t by all possible strings of $t - 1$ +'s and −'s, leading to $(16 - t) \cdot 2^{t-1}$ as the total number of such sequences.

For $t = 1$, we merely select one of the 15 distinct segments. For $t = 2$, there is an identification of segment 1 with segment 15. From any of the other 13 segments, + indicates that the second term of the sequence is the next higher number, while − indicates that it is the next lower number. From segment 1/15, + gives the sequence 1,2 while − gives the sequence 15,14. For $t = 3$, there is an identification of segment 1 with segment 14, and of segment 2 with segment 15. For any starting number from 3 to 13, inclusive, each of the patterns + +, + −, − +, − − yields a different sequence of length three, in a normal way. TABLE 1 shows the unique interpretation of these four patterns from the starting values 1/14 and 2/15. The situation for $t = 3$ is similarly summarized in TABLE 2. The "snake" for this case is illustrated in FIGURE 1. More generally, starting points for sequences are "distinct" if and only if they are distinct modulo $16 - t$. The reader is invited to fill in the remaining details.

Start \ Pattern	+ +	+ −	− +	− −
1/14 2/15	1,2,3 2,3,4	14,15,13 2,3,1	14,13,15 2,1,3	14,13,12 15,14,13

TABLE 1

Start \ Pattern	+ + +	+ + −	+ − +	+ − −
1/13 2/14 3/15	1,2,3,4 2,3,4,5 3,4,5,6	13,14,15,12 2,3,4,1 3,4,5,2	13,14,12,15 2,3,1,4 3,4,2,5	13,14,12,11 14,15,13,12 3,4,2,1

Start \ Pattern	− + +	− + −	− − +	− − −
1/13 2/14 3/15	13,12,14,15 2,1,3,4 3,2,4,5	13,12,14,11 14,13,15,12 3,2,4,1	13,12,11,14 14,13,12,15 3,2,1,4	13,12,11,10 14,13,12,11 15,14,13,12

TABLE 2

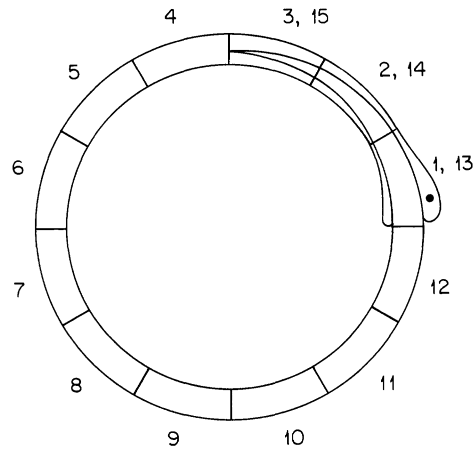

FIGURE 1

Some history

I first heard the original problem in 1952, when I was a graduate student at Harvard, by which time it already seemed to have the status of a "folk theorem." It was used in the 1965 Putnam Competition [**1**], where the published solution uses mathematical induction (yet another approach!), and the comment is made that "Several counting techniques were used by the contestants and many were quite ingenious." (From this it follows that "several $\geqslant$ many.") Equivalent formulations of the problem also appear in books on combinatorial analysis by Liu [**2**] and by Tucker [**3**]. Tucker asks the reader to find a recursion relating the number of solutions for the case $n+1$ to the number of solutions for the case n. The generalization presented above appears to be new, and represents yet another way to solve the original problem.

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References

[1] 26th William Lowell Putnam Competition, November 20, 1965. See Amer. Math. Monthly, 73 (1966) 728, 730.
[2] C. L. Liu, Introduction to Combinatorial Mathematics, McGraw-Hill, 1968, p. 23, exercise 1–25.
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Comments on the Cover Illustration: Torus with Complete 7-Map

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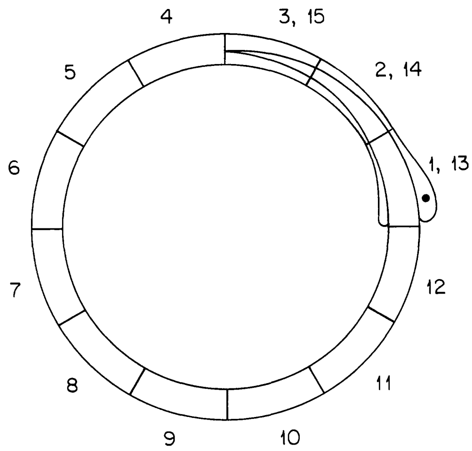
The cover illustration shows four views of a torus covered by a map of seven hexagonal regions, every two of which are adjacent. Two complementary techniques were used to render these images: ray-casting and parametric patches. The first used the representation of the torus as a quartic surface $T(x, y, z) = 0$. At each point (x, y), or pixel, of the screen a line through the point parallel to the z-axis is intersected with the surface. This yields a quartic equation in z, whose



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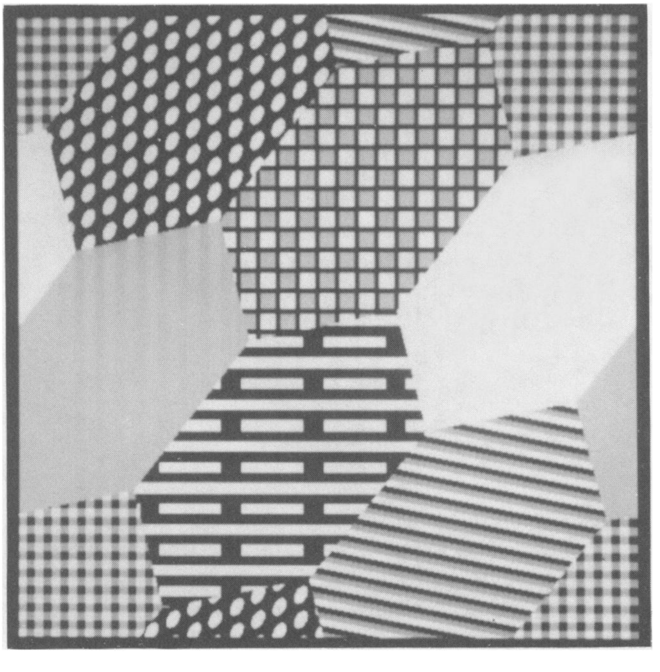
Seven grey scales would be hard for a viewer to distinguish, so more distinctive textures were created on the unit square (see photo below of the unrolled map). Once the ray-casting algorithm has determined a point (x, y, z) on the torus, it is necessary to determine the underlying "map color" at that point. This requires that we "unroll" the parametrization from its imbedded home in R^3 back into the (u, v) parameter space, the unit square $I \times I$. Once back in (u, v) space, linear algebra enables us to figure out which hexagon we are in. Each hexagon has its own pattern function, which takes a (u, v) pair and gives a shade value. This value is then used along with the lighting information at the point to color the point (x, y) on the display screen. (We note that this technique of inverting a patch mapping allows arbitrary patterns to be mapped onto the surface, and is widely used in the generation of synthetic computer images.)

Finally, some explanation of the four different images of the torus map shown on the cover is in order. After we trace back from (x, y, z) into (u, v) space, a different view of the torus can be easily obtained by using $(u, v + k)$ to determine our coloration (where of course $v + k$ is taken modulo 1). The four images correspond to $k = 0$, .25, .5, .75. Geometrically, this amounts to "rolling" the torus by quarter turns around its main circular axis, so that the smaller inner longitudes are brought up top and to the outside in turn.



Symmetry Groups of Knots

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The theory of knots, like many other branches of mathematics, had its origin in a physical situation in the real world, namely, the consideration of tangled loops of string. However, due to the tendency towards increasing abstraction, the theory has become more and more divorced from physical reality—and almost unintelligible to anyone working in a different subject, or even in another branch of mathematics.

The purpose of this note is to show that interesting, and not entirely trivial, problems exist at the "grass-roots" level. We shall be considering a strictly geometrical problem in three-dimensional space, a problem which arose from perusal of the knots in the well-known classic of practical knotting, the *Ashley Book of Knots* [**1**]. Some of the illustrations (FIGURE 1) suggest that the symmetry groups of knots can be of many kinds, but caution is necessary. For example, the knot in the right half of FIGURE 1 looks, at first sight, as if its symmetry group is $[3^+,4]$ or $[3,3]^+$ (see below for the identification of these groups). As we shall show shortly, this is not so, and in fact the groups $[3^+,4]$ and $[3,3]^+$ are *not* symmetry groups of *any* knot. The question naturally arises as to which groups can occur in this context; the purpose of our note is to give a complete answer to this problem.

It must be emphasized that here we are considering **symmetry groups**, that is, groups of isometries (distance-preserving transformations) of E^3 that map the knot onto itself, and not the various topological groups associated with knots that have been studied extensively over many years. (For a survey of these, and bibliography, see [**3**]. Since this paper was written, the interesting book [**5**] has appeared, which contains many illustrations of symmetric knots and links.)

To begin with, first we must define what we mean by a **knot**. From the point of view of this note the reader will not be led astray if he thinks of it as a piece of string that has been tangled and then its ends joined to form a continuous loop. Those who wish to be more formal should regard it as a simple embedding of a closed loop into E^3 with some local finiteness and smoothness properties. Various such properties have been proposed, any of which is appropriate in this case: we may insist that the embedding is PL (piecewise linear), or that some two-dimensional projection of the knot has only a finite number of crossings, or that there exists a $\delta > 0$ such that the intersection of every solid δ-sphere with the loop is either empty or a connected set. By a **plane knot** we mean any knot which lies entirely in a plane; such knots are, of course, trivial since they are not knotted, in the usual sense of the word.

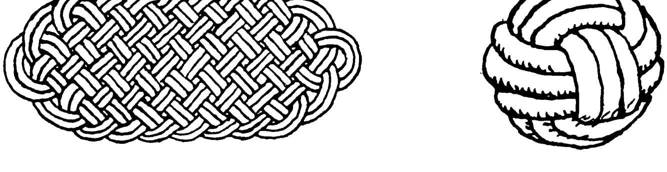


FIGURE 1. **Some knots from the *Ashley Book of Knots* [1].**

It is clear that the symmetry group $S(K)$ of a knot K must be a group of isometries of E^3 that leave fixed one point, namely, the centroid of the knot K. Throughout, this point will be denoted by the letter O. It is easy to see that only one continuous (that is, non-discrete) group of symmetries can arise. It is $[2, \infty]$, the infinite dihedral group with reflections in an "equatorial" plane and in all planes of "meridians". This is the group of a plane knot, namely of a circular loop lying in a plane. Otherwise the group is discrete, and we may refer to lists of such groups (see, for example, Coxeter & Moser [**2**] or Grünbaum & Shephard [**4**]) for a description of all the possibilities. It is well known that there are just 14 kinds of such groups denoted (in the above references) by $[q], [q]^+, [2, q], [2, q]^+, [2, q^+], [2^+, 2q], [2^+, 2q^+], [3, 3], [3, 3]^+, [3, 4], [3, 4]^+, [3^+, 4], [3, 5], [3, 5]^+$. Here q is a positive integer parameter. For the benefit of readers to whom this notation is not familiar we show, in FIGURE 2, spherical patterns illustrating all these groups.

Our main result is as follows:

THEOREM. *The only fixed-point groups of isometries of* E^3 *that can occur as symmetry groups of knots are* $[q], [q]^+, [2, q]^+, [2^+, 2q], [2^+, 2q^+]$ *and* $[2, \infty], [2, q], [2, q^+]$, *where* q *is a positive integer. The group* $[2, \infty]$ *can only occur in the case of a plane knot, and the same is true of* $[2, q]$ *and* $[2, q^+]$ *if* $q \geq 2$.

It should be observed that for small values of q repetitions occur in the list of groups, namely $[2, q]^+$ and $[2, q^+]$ with $q = 1$ are the same as $[q]^+$ with $q = 2$, $[2, q]$ with $q = 1$ is the same as $[q]$ with $q = 2$, and $[2^+, 2q]$ for $q = 1$ is the same as $[2, q^+]$ for $q = 2$. These repetitions are eliminated from FIGURE 2.

The fact that the groups listed in the theorem can actually occur as symmetry groups of knots should be apparent from examination of FIGURES 3 and 4. The proof that no other groups can occur is conveniently split into two parts. These relate to the existence, in $S(K)$, of rotational symmetries and of reflective symmetries. All axes of rotation and planes of reflection necessarily pass through the point O.

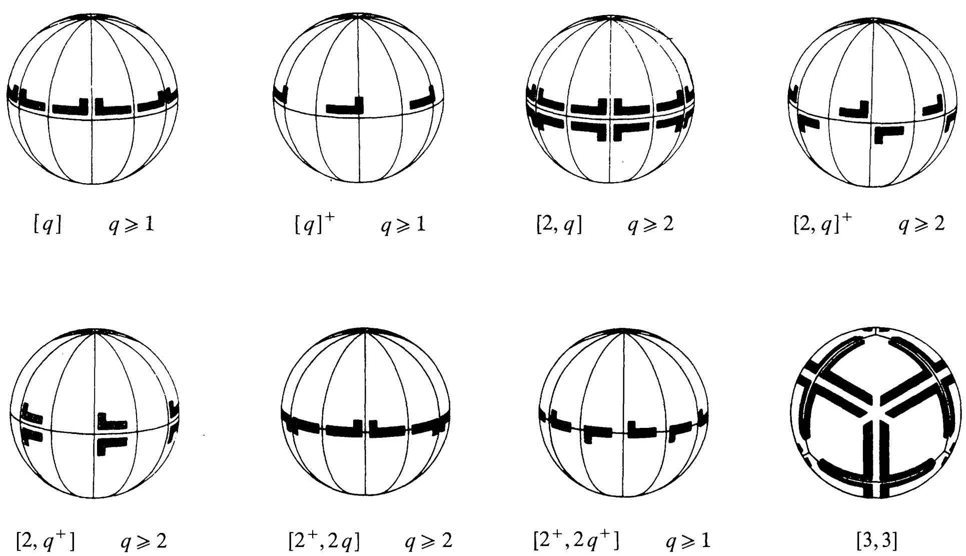
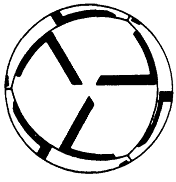
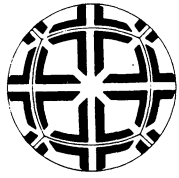


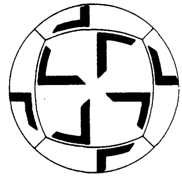
FIGURE 2. **The seven discrete groups and seven infinite families of isometries in E^3 that leave at least one point fixed. Each group is illustrated by a spherical pattern whose symmetry group is the group under consideration. Each of the symbols for the families of groups ($[q]$, $[q]^+$, $[2,q]$, $[2,q]^+$, $[2,q^+]$, $[2^+,2q]$, $[2^+,2q^+]$) contains a parameter q which takes positive values in the range indicated under the corresponding patterns. The diagrams show the case $q = 6$ except for the group $[2^+, 2q]$ where the parameter has the value $q = 3$.**



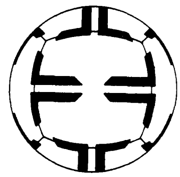
$[3,3]^+$



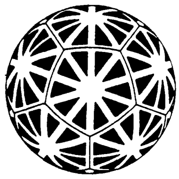
$[3,4]$



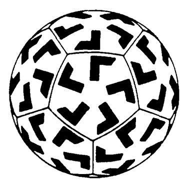
$[3,4]^+$



$[3^+,4]$



$[3,5]$



$[3,5]^+$

FIGURE 2. **(continued)**

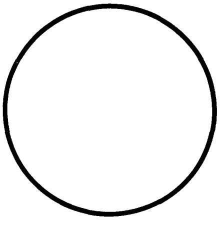
LEMMA 1. *It is impossible for the symmetry group $S(K)$ of a knot K to contain two or more distinct axes of 3-fold rotational symmetry.*

For suppose t_1, t_2 represent 3-fold rotations with axes a_1 and a_2. Let P_1 be any point on the loop and write $t_1 P_1 = P_2$, $t_1 P_2 = P_3$. Then the points P_1, P_2, P_3 must be separated on the loop by distances $L/3$, where L is the total length of the loop. Further, P_1, P_2, P_3 are the vertices of an equilateral triangle which defines a plane perpendicular to a_1. In a similar way let $t_2 P_1 = P_2'$ and $t_2 P_2' = P_3'$. Then either $P_2' = P_2$ and $P_3' = P_3$, or $P_2' = P_3$ and $P_3' = P_2$. In either case the planes $P_1 P_2 P_3$ and $P_1 P_2' P_3'$ coincide and hence the axes a_1, a_2, which are perpendicular to these planes and pass through 0, necessarily coincide as well. The lemma is therefore proved.

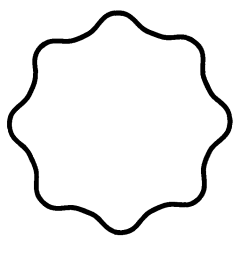
We deduce immediately that seven of the fourteen kinds of groups, namely $[3,3]$, $[3,3]^+$, $[3,4]$, $[3,4]^+$, $[3^+,4]$, $[3,5]$ and $[3,5]^+$ cannot occur as symmetry groups of knots.

LEMMA 2. *If a knot K is not a plane knot, all the planes of reflective symmetry of K must be collinear, that is, pass through a fixed line.*

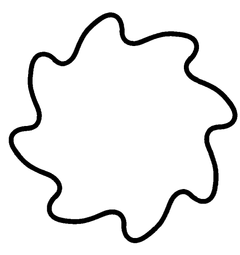
To begin with, we remark that if e is a plane of reflective symmetry of K, then the loop either lies entirely in e (it cannot escape!) and so K is a plane knot, or e cuts K perpendicularly. In the former case $S(K)$ can be the continuous group $[2,\infty]$ mentioned above or, as is easily seen, any of the groups $[2,q]$ or $[2,q^+]$ $(q \geqslant 2)$, see FIGURE 3. In each case the loop here is "unknotted". (If



(a)



(b)



(c)

FIGURE 3. **Examples showing the three kinds of symmetry groups possible in the case of a plane knot: (a) $[2,\infty]$, (b) $[2,q]$ and (c) $[2,q^+]$. In the examples shown in (b) and (c) the value of q is 8. In each case, the plane containing the knot is a plane of reflective symmetry.**

the loop cuts the plane perpendicularly then it is almost trivial that the number of intersections must be exactly two.)

Now suppose that K is not a plane knot and that there exist three planes e_1, e_2, e_3 of reflective symmetry of K which are not collinear. These meet in the point O and partition a 2-sphere centered at O into eight spherical triangles; denote one of the triangles by T and its vertices by V_1, V_2 and V_3. There are two cases:

(i) *At most one of the angles of T is a right angle*. Checking through the list of possible groups we see that in only three cases is it possible to find three planes which lead to a triangle T with the stated property, namely, for the groups [3,3], [3,4] and [3,5]. However, we already know that these cannot occur as symmetry groups of knots, so this case does not arise.

(ii) *At least two of the angles of T are right angles*. Suppose that the right angles are at V_2 and V_3, lying on the plane e_1. The planes e_2, e_3 meet at an angle equal to that of T at V_1 and, if the group is to be discrete, this must be a rational multiple of π. In fact, if it is $\pi m/n$, where m, n are integers and m/n is in its lowest terms, reflections in e_2 and e_3 generate a group of order $2n$ ($n \geqslant 2$). This group contains n planes of reflection, namely, e_2, e_3, and the images of these planes under repeated reflections; they cut a "star" of n lines on e_1. As the knot does not lie in any of these planes and necessarily cuts e_1, it is possible to choose a point P which lies on the loop and on e_1, but not on any of the n lines of the star. Then successive reflections in e_2, e_3 lead to a set of $2n$ points $P_1, P_2, \ldots, P_{2n}$ which are the vertices of a $2n$-gon lying in a plane perpendicular to OV_1 (that is, to the line of intersection of e_2 and e_3). But the fact that e_1 is perpendicular to this

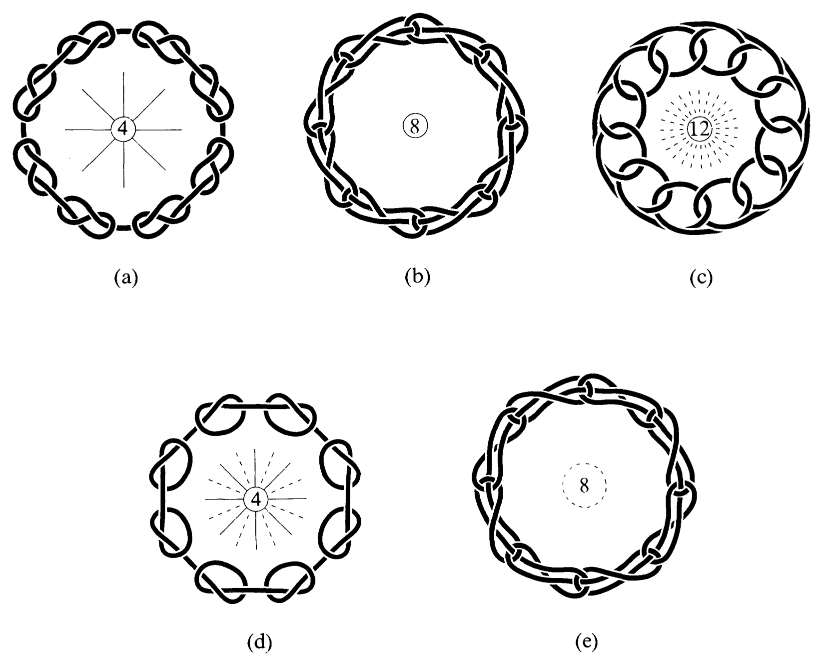


FIGURE 4. Examples showing the five kinds of symmetry groups possible for knots which are not plane: (a) $[q]$, (b) $[q]^+$, (c) $[2,q]^+$, (d) $[2^+,2q]$ and (e) $[2^+,2q^+]$. In the examples shown, q takes the value 4 except in the case of (b) where $q = 8$ and (c) where $q = 12$. Elements of the symmetry groups are schematically indicated as follows. A central circle with an attached number m means that there is an axis of m-fold rotational symmetry. A solid line indicates a reflection; it is the intersection of a plane of reflective symmetry with the plane of the paper. A dashed line denotes an axis of 2-fold rotational symmetry. A dashed circle to which a number m is attached means a rotary reflection of period m.

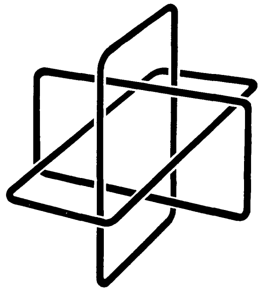


FIGURE 5. **A link of three loops with symmetry group $[3^+, 4]$. There are three mutually perpendicular planes of reflection, each containing one of the loops, three mutually perpendicular axes of 2-fold rotation, each meeting two of the loops, and four axes of 3-fold rotation.**

line and contains P_1 shows that it must contain all the other points P_i $(1 \leqslant i \leqslant 2n)$. This is impossible since, as we have remarked, the fact that K is not a plane knot implies that it can meet the plane e_1 in at most two points, and $2n \geqslant 4$. This contradiction shows that the second case also cannot arise, and so the lemma is proved.

Lemma 2 shows that for $q \geqslant 2$ groups $[2, q]$ and $[2, q^+]$ cannot occur as symmetry groups of non-plane knots. With the seven groups eliminated by Lemma 1 this leaves us with just the groups given in the statement of the Theorem. The proof of the Theorem is therefore completed.

We conclude with some general remarks. The above proof depends essentially on the fact that a single simple loop is under consideration. In the case of a *link* of two or more loops the proof fails; Lemma 1 fails in the case of a link with $3k$ loops ($k \geqslant 1$) and Lemma 2 fails for any number of loops greater than 1. This means that the variety of possible symmetry groups is greater in the case of links. Clearly the actual groups that can occur depend essentially on the number of loops in the link; in fact, every group can be realized as a symmetry group of a link if the number of loops in the link is not prescribed. In FIGURE 5 we show a link with three loops whose symmetry group is $[3^+,4]$. All three loops are interlinked, but if any one of them is removed then the other two can be separated (that is, they are not linked together); they are sometimes known as the Borromean rings. It would be interesting and, we believe, not difficult, to determine all possible groups for all links with prescribed numbers of loops. In particular, the links in which the group of symmetries acts transitively on the loops appear to be deserving of attention. Several remarkable examples are shown in [**5**].

Another problem is to decide if there is any relationship between the possible symmetry groups of a particular knot and the topological invariants of that knot. The great majority of knots appear to admit no embeddings with other than the trivial symmetry group. In this connection it is worth noting that each of the knots shown in FIGURE 4 is *maximal* regarding symmetry. By this we mean that no knot of the same type (in a topological sense) can have more symmetries than that shown.

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Differentiability and the Arc Chord Ratio

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Consider a simple closed rectifiable curve, C, in the plane. For points p and q on C, $l(p,q)$ denotes the smallest positive arc length along C from p to q and $d(p,q)$ denotes the usual Euclidean distance from p to q. Since arc length is measured by a limiting process using inscribed polygonal approximations, we might assume that sufficiently small sides of such polygons closely approximate the length of the part of the curve they subtend. In other words, we might expect that the **arc chord ratio,** $l(p,q)/d(p,q)$, will be close to 1 when p and q are close to one another.

A provocative first example is provided by a square. Although for any fixed p

$$\lim_{q\to p}\frac{l(p,q)}{d(p,q)}=1,$$

we can still find p and q arbitrarily close to each other with $l(p,q)/d(p,q)=\sqrt{2}$ —merely take p and q equidistant from a corner. Thus the arc chord ratio need not approach 1 in the neighborhood of a corner if we allow both points to vary in that neighborhood. The same difficulty arises on any curve for which there are points where one-sided derivatives exist and are not equal.

How do conditions on the differentiability of the curve interact with the behavior of the arc chord ratio? We shall show that continuous differentiability implies that the arc chord ratio can be made arbitrarily close to 1 by making $d(p,q)$ sufficiently small. The converse is false. The connection between the behavior of the arc chord ratio and differentiability was suggested in a conjecture made by Herda [**1**]. We will resolve his conjecture.

The reader, accustomed to avoiding vertical tangents, should notice that any smooth closed curve will have vertical tangents—by **smooth** we mean curves given by parametrizations of the form $a(t)=(x(t),y(t))$ with coordinate functions $x(t)$ and $y(t)$ continuously differentiable.

Smooth is sufficient, differentiability is not

It is known and is occasionally taught in courses on advanced calculus ([**4**], page 245) that *if C is smooth and p is any fixed point on C, then*

$$\lim_{q\to p}\frac{l(p,q)}{d(p,q)}=1.$$

We extend this by showing that the arc length can be made uniformly close to the Euclidean distance. Our proof illustrates the use of compactness to extend a local property to a global one.

Theorem 1. *If the closed rectifiable curve C is continuously differentiable, then for all p, q on C,*

$$\lim_{\epsilon\downarrow 0}\ \operatorname*{lub}_{d(p,q)\leqslant\epsilon}\frac{l(p,q)}{d(p,q)}=1.$$

Proof. First we establish that for any $\delta>0$, C can be covered by neighborhoods in which $l(p,q)/d(p,q)<1+\delta$. Given any point O on C, there exists a parametrization of C by arc length s measured from O. Form a neighborhood N of $O=a(0)$ consisting of points for which the unit tangent vector, $\mathbf{T}(s)=(\cos\psi(s),\sin\psi(s))$, is close to $\mathbf{T}(0)$, i.e., $\psi(s)\in(\psi(0)-\eta,\psi(0)+\eta)$, where $\psi(s)$ is the angle between the x-axis and the tangent and η can be chosen to suit our convenience. For any two points p and q in N, we shall compare the lengths of segments in a polygonal arc from p to q with their projections on the line segment from p to q. Consider an arbitrary polygonal approximation to the arc from p to q, made up of straight line segments connecting the successive points $C_0, \dots, C_n$. Each segment C_jC_{j+1} can be projected onto the vector from p to q. If this projection is D_jD_{j+1}, then we can compare the lengths $d(C_j, C_{j+1})$ and $d(D_j, D_{j+1})$. Indeed, a bit of trigonometric fiddling indicates that if $\eta < \operatorname{arcsec}(1+\delta)/2$ then $d(C_j, C_{j+1}) < d(D_j, D_{j+1})(1+\delta)$. Since $l(p,q)$ is the least upper bound of the sums of the lengths $d(C_j, C_{j+1})$ and the sum of their projections on the vector from p to q is just $d(p,q)$, we see that $l(p,q) \leqslant d(p,q)(1+\delta)$ for any points p and q in N.

Now we show that there is an ϵ for which $d(p,q) < \epsilon$ implies $l(p,q)/d(p,q) < 1+\delta$. We transfer the problem in the obvious way to the compact set given by the real numbers modulo L, where L is the length of C. Compactness implies that there is a finite cover of $[0, L]$ by nice neighborhoods $N_1, \dots, N_n$ as created in the previous paragraph. Assume (by shrinking the neighborhoods if necessary) that $N_j = (a_j, b_j)$ with $a_j < a_{j+1}$ and $b_j < b_{j+1}$ and that the only neighborhoods which overlap are adjacent. Let m be the length of the minimum overlap. Every point p in $[0, L]$ then must have its $m/2$ neighborhood wholly contained in some N_j so that $d(p,q) < m/2$ implies that $l(p,q)/d(p,q) < 1+\delta$. (The existence of this uniform $\epsilon = m/2$ is guaranteed in general by the Lebesgue Number Lemma, [3].) This completes the proof of Theorem 1.

Here is an example of a differentiable but not continuously differentiable curve for which the arc chord ratio does not approach 1. The curve, shown in FIGURE 1, consists of an infinite sequence of smoothly connected semicircles. The semicircles have diameters given by the segment from $(1/(n+1), 0)$ to $(1/n, 0)$ and lie alternately above and below the x-axis. The curve can be closed with slope zero at the origin since the right hand limit of the slope of the segments joining the origin to points on the curve is zero. Suppose, for example, the origin $(0,0)$ is joined to (x, y) on the nth semicircle. The slope of this segment is then less than the quotient of the maximum y on the semicircle, which is $1/2n(n+1)$, and the minimum x on this semicircle, which is $1/2n$, so this slope is less than $1/(n+1)$. As the point (x, y) approaches the origin, n gets larger and so the slope approaches zero. Note, however, that the curve is not continuously differentiable at the origin.

The curve in FIGURE 1 is rectifiable since the string of semicircles has its total length bounded by $\pi/2$ times the sum of the diameters of the semicircles. The arc chord ratio $l(p,q)/d(p,q)$ equals $\pi/2$ whenever p and q are endpoints of the diameter of one of the semicircles. Since for these p and q, $d(p,q)$ can be made arbitrarily small, the limit in Theorem 1 cannot possibly be 1. In fact, one can show that for $p = (0,0)$ and q any point on the sequence of semicircles, the arc chord ratio is greater than $\pi/3$.

The curve in FIGURE 1 provides a counterexample to Herda's conjecture. This conjecture employs a subtly different condition on the arc chord ratio from the one in Theorem 1.

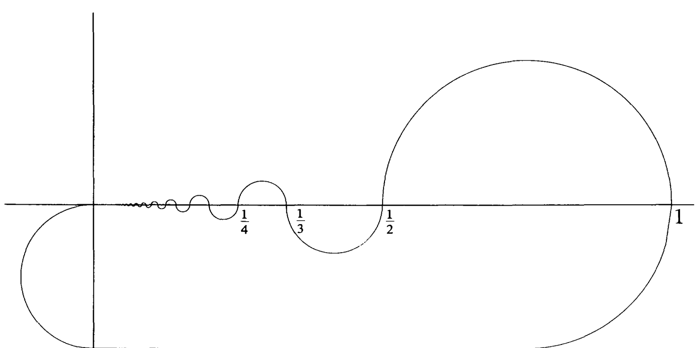

FIGURE 1

HERDA'S CONJECTURE. *C is differentiable iff*

$$\operatorname*{glb}_{t>0} \operatorname*{lub}_{d(p,q)=t} \frac{l(p,q)}{d(p,q)} = 1.$$

The careful reader will note that Theorem 1 assumes smoothness and proves a slightly different arc chord condition—a stronger condition, in fact. Theorem 1 tells us that if C is continuously differentiable, then the arc chord ratio "stays" close to 1. Existence of the limit in Theorem 1 implies the arc chord condition in Herda's conjecture. An example for which the two concepts differ is indicated in FIGURE 2. The curve is essentially a lozenge with very occasional, in fact, exponentially scarce smooth spikes reaching up to the line $y = x$. For $n > 2$, let $x_n = n^{-3n}$ and $r_n = n^{-(3n+1)}$. The sides of each spike are formed from vertical line segments at $x_n \pm r_n$ and are connected together and to the x-axis by half and quarter circles, respectively, of radius r_n. These spikes are connected to one another by segments of the x-axis, and the curve is closed by 2 semicircles and a straight line segment (the sides and bottom of the lozenge). Of course FIGURE 2 is not to scale—we cannot depict what a miniscule ripple the $n+1$st spike makes in the line from the nth spike to the origin. Careful checking of the constituent parts of the curve indicates that, as t approaches zero, the least upper bound of the arc chord ratio for chord length equal to t oscillates "down" ever nearer to 1 (for $t = n^{-(3n+2)}$) and "up" above $\pi/2$ (for $t =$ the diameter of our spikes). This oscillation means that the arc chord condition in Herda's conjecture is satisfied, yet the arc chord condition in Theorem 1 is not.

Our colleague, David Lesley, has pointed out that spirals provide a counterexample to the converse of Theorem 1. Here is an example of a curve which is not differentiable at the origin, but for which

$$\lim_{\epsilon \downarrow 0} \operatorname*{lub}_{d(p,q) \leqslant \epsilon} \frac{l(p,q)}{d(p,q)} = 1.$$

This shows that the other half of Herda's conjecture is also false. Our curve, C, is formed from three parts. We begin with the polar curve

$$r(\theta) = e^{-\theta^2}$$

with $0 \leqslant \theta \leqslant \infty$ (the origin corresponds to $\theta = \infty$, where the curve will not be differentiable). We add a 180-degree rotation of this exponential spiral. The third piece can be any smooth connection of the two spirals at their outer endpoints (where $\theta = 0$). The resulting simple closed curve is clearly not differentiable at the origin but is smooth everywhere else; see FIGURE 3.

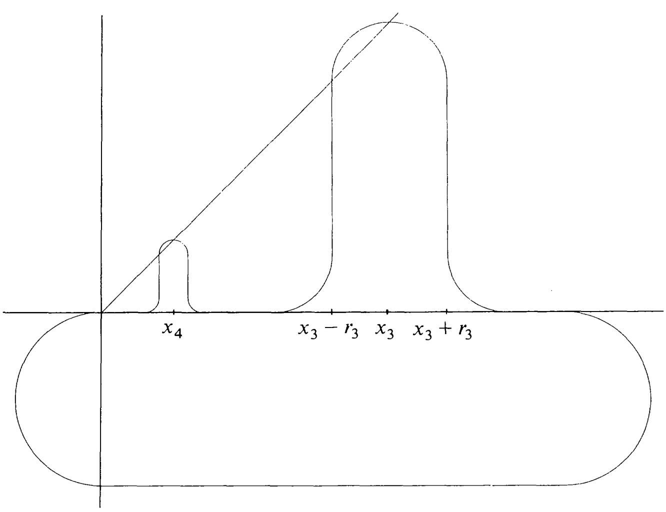

FIGURE 2

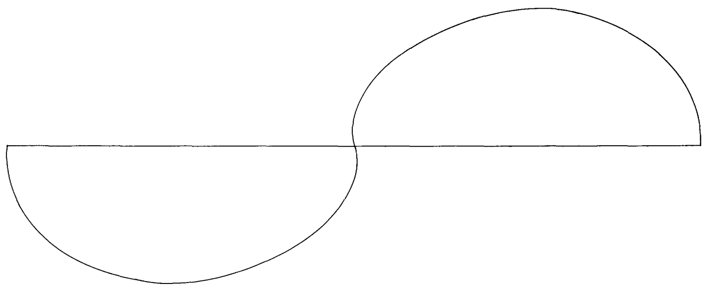

FIGURE 3. **The spiral $r = e^{-\theta^2}$ and its reflection.**

Although this curve can be parametrized by directed arc length from any point on it, computation is easier if we use the following parametrizations of the spirals:

$$a(t) = \left(e^{-t^2}\cos t, e^{-t^2}\sin t\right),$$
$$b(s) = \left(e^{-s^2}\cos(s+\pi), e^{-s^2}\sin(s+\pi)\right) = -a(s).$$

Rectifiability of $a(t)$ follows from the fact that the improper integral for arc length is dominated by the improper integral of the derivative of $-2e^{-t^2}$. This establishes an upper bound for the length of any polygonal approximation. It follows that the whole curve is rectifiable, assuming a smooth connection of a $a(0)$ to $b(0)$.

Now we must establish that the arc chord ratio approaches 1. Since C is smooth except at the origin, Theorem 1 indicates that the only difficulty arises in considering two points which are given by $q = a(t)$ and $p = b(s)$ (N.B. the origin is covered since we employ the corruption $b(\infty)$). Using the inequalities

$$\sqrt{1+4x^2} < \frac{1}{4x} + 2x < \left(1 + \frac{1}{8t^2}\right)2x \tag{1}$$

for $0 < t < x$, we note, with $\mathbf{O} = (0,0)$, that

$$\begin{aligned} l(a(t),\mathbf{O}) &= \int_t^\infty e^{-x^2}\sqrt{1+4x^2}\,dx \\ &< \left(1+\frac{1}{8t^2}\right)\int_t^\infty 2xe^{-x^2}\,dx \\ &= \left(1+\frac{1}{8t^2}\right)e^{-t^2} \end{aligned} \tag{2}$$

Thus the arc chord ratio for $a(t)$ (or for $b(t)$) and the origin is dominated by $(1 + 1/8t^2)$ which approaches 1 as t approaches ∞. For two points $a(t)$ and $b(s)$,

$$\begin{aligned} \frac{l(a(t),b(s))}{d(a(t),b(s))} &\leqslant \frac{l(a(t),\mathbf{O})}{d(a(t),b(s))} + \frac{l(\mathbf{O},b(s))}{d(a(t),b(s))} \\ &= \frac{d(a(t),\mathbf{O})}{d(a(t),b(s))}\frac{l(a(t),\mathbf{O})}{d(a(t),\mathbf{O})} + \frac{d(\mathbf{O},b(s))}{d(a(t),b(s))}\frac{l(\mathbf{O},b(s))}{d(\mathbf{O},b(s))} \\ &< \frac{d(a(t),\mathbf{O})}{d(a(t),b(s))}\left(1+\frac{1}{8t^2}\right) + \frac{d(\mathbf{O},b(s))}{d(a(t),b(s))}\left(1+\frac{1}{8s^2}\right) \\ &\leqslant \left[\frac{d(a(t),\mathbf{O}) + d(\mathbf{O},b(s))}{d(a(t),b(s))}\right]\left(1+\frac{1}{8t^2}\right). \end{aligned} \tag{3}$$

Now let
$$f(x) = \underset{d(a(t),b(s))\leqslant x}{\text{lub}} \frac{l(a(t),b(s))}{d(a(t),b(s))};$$

$f(x)$ is a weakly decreasing function of x which is always greater than or equal to 1. If the arc chord ratio does not approach 1, then we may assume there is some positive constant A such that $f(x) > 1 + A$ for all x. Then there exist increasing sequences s_n and t_n both approaching ∞ with

$$\frac{l(a(t_n), b(s_n))}{d(a(t_n), b(s_n))} > 1 + A \quad \text{for all } n. \tag{4}$$

In fact, we may assume that $t_n \leqslant s_n$ since exchanging the values will not affect the underlying geometric relationship. We consider two possibilities.

First suppose that some subsequence has the property that $s_n - t_n$ approaches 0 as n approaches ∞. Then the law of cosines yields

$$[d(a(t_n), b(s_n))]^2 = [d(a(t_n), \mathbf{O})]^2 + [d(\mathbf{O}, b(s_n))]^2 - 2\cos\alpha\, d(a(t_n), \mathbf{O})\, d(\mathbf{O}, b(s_n)),$$

where α is the angle at $\mathbf{O}$ in the triangle with vertices $\mathbf{O}$, $a(t_n)$ and $b(s_n)$. However, this angle is given by $s_n + \pi - t_n$ so it approaches π as n approaches ∞ and so

$$[d(a(t_n), \mathbf{O}) + d(\mathbf{O}, b(s_n))]^2 / [d(a(t_n), b(s_n))]^2 \to 1.$$

We see from (3) that the arc chord ratio must approach 1 in this case, and we have contradicted (4).

The second possibility is that no subsequence has $s_n - t_n$ approaching 0. In this case there is some positive c such that ultimately $s_n - t_n > c$. We will use this in a refined version of our inequality (3). Since $a(t)$ and $b(s)$ lie on concentric circles about the origin and have radii e^{-t^2} and e^{-s^2}, respectively, the distance between the closest points on these two circles is $e^{-t^2} - e^{-s^2}$, which must be less than $d(a(t), b(s))$. This allows us to deduce from (3) that

$$\frac{l(a(t), b(s))}{d(a(t), b(s))} \leqslant \left[\frac{e^{-t^2} + e^{-s^2}}{e^{-t^2} - e^{-s^2}}\right]\left(1 + \frac{1}{8t^2}\right)$$
$$= \left[\frac{1 + e^{t^2 - s^2}}{1 - e^{t^2 - s^2}}\right]\left(1 + \frac{1}{8t^2}\right). \tag{5}$$

But $t_n^2 - s_n^2 < t_n^2 - (t_n + c)^2 = -2ct_n - c^2$ and so

$$\lim_{n \to \infty} e^{t_n^2 - s_n^2} = 0.$$

Thus the arc chord ratio must approach 1 as n approaches ∞.

We conclude by noting another connection with spirals. If a curve C has constant arc chord ratio with respect to some point on C (which we take as the origin), then C must be contained in an Archimedean spiral. To see this, suppose that C is given in polar form $r(t)$ with t representing angular measure as usual, and suppose that $r = 0$ gives the origin $\mathbf{O}$. There arises naturally the parametrization $a(t) = (r(t)\cos t, r(t)\sin t)$, with $a(0) = \mathbf{O}$. Let $l(t)$ represent the arc length from $\mathbf{O}$ to $a(t)$. Differentiate the expression $l(t) = Kr(t)$ and replace $l'(t)^2$ by $r'(t)^2 + r(t)^2$. Solving for $r'(t)/r(t)$ and integrating yields $r(t) = ae^{bt}$, for a and b constants.

Consideration of other arc chord conditions can be found in [**2**] and its references.

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Uncountable Fields Have Proper Uncountable Subfields

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In a first encounter of the study of fields in an abstract algebra course, a student learns about various subfields of $\mathbb{R}$ and $\mathbb{C}$ which are usually obtained by adjoining a finite number of elements to the field $\mathbb{Q}$. The student may also learn about the subfield of $\mathbb{R}$ consisting of those real numbers which are algebraic over $\mathbb{Q}$. Each of these subfields of $\mathbb{R}$ has the property that it contains a countable number of elements. Since the field $\mathbb{R}$ is uncountable, the question naturally arises as to whether there exists an uncountable proper subfield of $\mathbb{R}$. An affirmative answer to this may be given by appealing to the existence of a transcendental basis of $\mathbb{R}$ over $\mathbb{Q}$ [**1**]. The idea of a transcendental basis is not commonly encountered in an undergraduate abstract algebra course, but many students do encounter Zorn's lemma with which one can construct an elementary, yet interesting, proof of the following proposition.

PROPOSITION. *If all proper subfields of a field F contain a countable number of elements, then F contains a countable number of elements.*

To see this, we first note that the prime field of F, which is the smallest subfield of F (the field generated by the multiplicative identity), is isomorphic either to $\mathbb{Q}$ or to a finite field of p elements. So without loss of generality we may assume that F properly contains its prime field. Let c be an element of F which is not in the prime field. Define S to be the collection of all subfields of F which do not contain c. S is nonempty because the prime field is in S. Moreover, the collection S is partially ordered by set inclusion. If $\{K_\alpha\}$ is any chain from S, then $\bigcup_\alpha K_\alpha$ is in S and is an upper bound. By Zorn's lemma, there exists a maximal subfield M of F which does not contain c. By assumption M is countable. Let y be an arbitrary element of $F \setminus M$. Then $M(y)$ is a field which properly contains M and, hence, by the maximality of M, we have that c is in $M(y)$. This means that $c = p(y)/q(y)$, where $p(y)$ and $q(y)$ are polynomials in y with coefficients from M. We then obtain $p(y) - cq(y) = 0$, which shows that y is algebraic over the field $M(c)$. But $M(c)$ is countable and there are only countably many algebraic elements over $M(c)$. (The number of polynomials over a countable field is countable and each polynomial has only a finite number of roots in the algebraic closure of the field.) Consequently both M and $F \setminus M$ are countable, which gives us the fact that F is countable.

An immediate corollary to the proposition is that the uncountable field $\mathbb{R}$ contains an uncountable proper subfield.

A few remarks are worth noting. First, keeping the notation of the proof, we see that every element of the field F is algebraic over $M(c)$. Second, the proof is valid for any infinite cardinal number. That is, if $\aleph$ is any infinite cardinal number and if all proper subfields of F have cardinality at most $\aleph$, then F has cardinality at most $\aleph$. However, it is not true that the field must be finite if all proper subfields of the field are finite. A standard construction shows that there exists a countable field all of whose proper subfields are finite. To see this, let F_1 be the finite field of p elements for some prime p. There is always an irreducible polynomial of degree 2 over any finite field of q elements because the number of polynomials of the form $x^2 + ax + b$ is q^2 and the number of those of the form $(x-a)(x-b)$ is $q + q(q-1)/2$. Suppose we have defined fields $F_1, \ldots, F_{n-1}$ for some $n \geqslant 2$, such that each field is a subfield of the algebraic closure of F_1 and $F_1 \subset F_2 \subset \cdots \subset F_{n-1}$. Then let a_n be an element of the algebraic closure of F_1 which is a root of an irreducible polynomial in $F_{n-1}[x]$ and define $F_n = F_{n-1}[a_n]$. Then $F_1 \subset F_2 \subset F_3 \subset \cdots$ and $F = \bigcup_n F_n$ is a countable field. Assume that K is a countable subfield of F. For each α in K, there is a smallest $n(\alpha)$ such that α is in $F_{n(\alpha)}$. Since α is not in $F_{n(\alpha)-1}$, α is a generator of the cyclic group of nonzero elements in $F_{n(\alpha)}$. Hence we have $F_{n(\alpha)} \subset K$. Since there are countably many elements in K, $\sup\{n(\alpha)\colon \alpha \in K\} = \infty$. The nesting of the F_n's now shows that $F = \bigcup_\alpha F_{n(\alpha)} = K$.

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Imitation of an Iteration

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A most fascinating and frustrating problem is the **Collatz $3x+1$ problem**. *Does repeated iteration of the function*

$$T(n) = \begin{cases} \dfrac{n}{2}, & n \text{ even}, \\ \dfrac{3n+1}{2}, & n \text{ odd}, \end{cases}$$

always reach 1, *for any positive starting point*? This convergence to 1 has been verified for n up to the billions, but a proof of convergence for all n does not yet exist. In fact, the problem seems intractable as stated, and has a place of honor in R. Guy's article, "Don't Try to Solve These Problems!" [7]. What *is* known of this problem is probabilistic in nature, such as: "For almost all positive starting points, there is some iterate smaller than the starting point."

In this note we consider a generalization of the $3x+1$ problem and prove a fairly strong probabilistic result. This result is not completely new, but the purpose here is to demonstrate that a probabilistic model can give information concerning a number-theoretic problem. Finally, we make a strong conjecture concerning the generalized iteration problem, and present some empirical data concerning this conjecture.

We deal with the following generalization of the $3x+1$ problem. Define a **Collatz-type iteration** function $C(n)$ by its action on different residue classes of the positive integers mod d as

$$C(n) = \left\{ h_i(n) = \frac{a_i n + b_i}{d}, \quad \text{for } n \equiv i \bmod d\colon 0 \leqslant i \leqslant d-1 \right\}, \tag{1}$$

where $a_i n + b_i \equiv 0 \bmod d$. For example, the function $T(n)$ is given by definition (1) with $d=2$, and $a_0 = 1$, $b_0 = 0$, $a_1 = 3$, $b_1 = 1$. Let the trajectory of n be the sequence of iterations $(n, C(n), C^{(2)}(n), C^{(3)}(n), \ldots)$. We say that the **trajectory** of n **converges to a cycle** if the sequence ends in a repeating loop. For example, the trajectory of 13 for $T(n)$ in the $3x+1$ problem is $(13, 20, 10, 5, 8, 4, 2, 1, 2, \ldots)$, which converges to the cycle $(1,2,1)$. Clearly, the trajectory of n converges to $(1,2,1)$ in the $3x+1$ problem if and only if some iterate of $T(n)$ reaches 1, so we have another way of looking at the problem—by looking at convergence of trajectories.

With the notation in (1), we can now ask the general Collatz-type question: *For which functions $C(n)$ do the trajectories of all positive n converge to a finite set of known cycles*?

fields $F_1, \ldots, F_{n-1}$ for some $n \geqslant 2$, such that each field is a subfield of the algebraic closure of F_1 and $F_1 \subset F_2 \subset \cdots \subset F_{n-1}$. Then let a_n be an element of the algebraic closure of F_1 which is a root of an irreducible polynomial in $F_{n-1}[x]$ and define $F_n = F_{n-1}[a_n]$. Then $F_1 \subset F_2 \subset F_3 \subset \cdots$ and $F = \bigcup_n F_n$ is a countable field. Assume that K is a countable subfield of F. For each α in K, there is a smallest $n(\alpha)$ such that α is in $F_{n(\alpha)}$. Since α is not in $F_{n(\alpha)-1}$, α is a generator of the cyclic group of nonzero elements in $F_{n(\alpha)}$. Hence we have $F_{n(\alpha)} \subset K$. Since there are countably many elements in K, $\sup\{n(\alpha)\colon \alpha \in K\} = \infty$. The nesting of the F_n's now shows that $F = \bigcup_\alpha F_{n(\alpha)} = K$.

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We deal with the following generalization of the $3x+1$ problem. Define a **Collatz-type iteration** function $C(n)$ by its action on different residue classes of the positive integers mod d as

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where $a_i n + b_i \equiv 0 \bmod d$. For example, the function $T(n)$ is given by definition (1) with $d = 2$, and $a_0 = 1$, $b_0 = 0$, $a_1 = 3$, $b_1 = 1$. Let the trajectory of n be the sequence of iterations $(n, C(n), C^{(2)}(n), C^{(3)}(n), \ldots)$. We say that the **trajectory** of n **converges to a cycle** if the sequence ends in a repeating loop. For example, the trajectory of 13 for $T(n)$ in the $3x+1$ problem is $(13, 20, 10, 5, 8, 4, 2, 1, 2, \ldots)$, which converges to the cycle $(1, 2, 1)$. Clearly, the trajectory of n converges to $(1, 2, 1)$ in the $3x+1$ problem if and only if some iterate of $T(n)$ reaches 1, so we have another way of looking at the problem—by looking at convergence of trajectories.

With the notation in (1), we can now ask the general Collatz-type question: *For which functions $C(n)$ do the trajectories of all positive n converge to a finite set of known cycles?*

In this note we deal with a subset of Collatz-type iteration functions that contain the $3x+1$ problem and the following two examples.

$3x-1$ Problem. In this problem, the twin to the $3x+1$ problem, the iteration function is defined by

$$T^*(n)=\begin{cases}\dfrac{n}{2}, & n \text{ even},\\ \dfrac{3n-1}{2}, & n \text{ odd},\end{cases}$$

with known cycles (1), (5,7,10,5), and (17,25,37,55,82,41,61,91,136,68,34,17).

$4x\pm 1$ Problem. This problem is given by the iteration of

$$T^{**}(n)=\begin{cases}\dfrac{n}{3}, & n\equiv 0(\operatorname{mod}3),\\ \dfrac{4n-1}{3}, & n\equiv 1(\operatorname{mod}3),\\ \dfrac{4n+1}{3}, & n\equiv 2(\operatorname{mod}3),\end{cases}$$

and the only known cycle is (1).

To study the behavior of functions like these we make the following definitions. Let $R=\{r_1,r_2,\dots\}$ be the set of smallest members (to be thought of as generators) of the *known* cycles for a given Collatz-type iteration. For example, $R=\{1,5,17\}$ for the $3x-1$ problem, and $R=\{1\}$ for both the $3x+1$ and $4x\pm1$ problems. For each positive n, we define the **total stopping time**, $\sigma_\infty(n)$, to be the first k such that $C^{(k)}(n)$ belongs to R, if such a k exists (in our $3x+1$ example, $\sigma_\infty(13)=7$). If no such k exists, we set $\sigma_\infty(n)=\infty$. Therefore, a restatement of our question is: *Is* $\sigma_\infty(n)<\infty$ *for all positive* n?

It is very hard to study the function $\sigma_\infty(n)$ directly, but we can study "intermediate" functions. For $\lambda>0$, define the **λth stopping time** $\sigma_\lambda(n)$ to be the first k so that $C^{(k)}(n)$ is less than n/λ. Clearly, for $\sigma_\infty(n)<\infty$ to hold, we must have $\sigma_\lambda(n)<\infty$ for large n, so study of $\sigma_\lambda(n)$ is warranted.

To study $\sigma_\lambda(n)$ we will use the sequence of **(λ,k)-integers**, $S_\lambda(k)$, defined to be those integers whose λth stopping time is less than or equal to k. For an integer sequence A, we define the **density** of A, denoted δA, to be

$$\delta A=\liminf_{n\to\infty} A(n)/n,\text{ where } A(n)=|\{k\in A: k\leqslant n\}|.$$

With this notation, the statement that $\sigma_\lambda(n)<\infty$ for almost all n can be restated as

$$\lim_{k\to\infty}\delta S_\lambda(k)=1. \tag{2}$$

Our main result will be to show that (2) holds for any $C(n)$ satisfying certain conditions. To do this we develop a probabilistic model to imitate the action of $C(n)$.

Let a random variable X be defined with the following distribution:

$$P\left(X=\frac{a_i}{d}\right)=\frac{1}{d},\qquad i=0,1,2,\dots,d-1. \tag{3}$$

If some of the a_i are identical, we let the probabilities accumulate; for example, in the $4x\pm1$ problem, X is defined by

$$P\left(X=\frac{4}{3}\right)=\frac{2}{3},\qquad P\left(X=\frac{1}{3}\right)=\frac{1}{3}.$$

This probability model will be useful for a special kind of Collatz-type function, which we now will define. The function $C(n)$ (or the problem generated by $C(n)$) is **mixing** if, for each function $h_i(n)$, defined by (1), and any $k \bmod d^m$, there is an $n^* \bmod d^{m+1}$ so that

$$h_i(n^*) \equiv k \bmod d^m.$$

The idea is that each of the $h_i(n)$ distributes evenly among all residue classes (mod d). We note that the $3x+1$, $3x-1$, and $4x \pm 1$ problems are all mixing, as simple congruence relations show. In a similar manner, it can be shown that a sufficient (but not necessary) condition for a Collatz-type problem to be mixing is that a_i and d are relatively prime for all i.

Mixing Collatz-type iteration problems are precisely those for which the probabilistic model yields information. Let $X_1, X_2, \ldots$ be independent, identically distributed copies of the random variable X, defined by (3). For $\lambda > 0$, define a new random variable $U(\lambda)$ to be the first k for which $X_1 X_2 \cdots X_k < 1/\lambda$, if such a k exists; otherwise, we set $U(\lambda) = \infty$.

If we look at the Collatz-type iteration, we see that for large n, $C(n)$ is approximately $(a_i/d)n$, if $n \equiv i \bmod d$. So for a "typical" large n, the trajectory of n is imitated by the sequence $(n, nX_1, nX_1X_2, \ldots)$, and we therefore think of the function U as counting the "number of steps it takes for n to get small." The relation between $U(\lambda)$ and the Collatz iteration problem is the following.

THEOREM 1. *Let $C(n)$ be mixing, and let $\lambda > 0$. Then*

$$P(U(\lambda) \leqslant k) = \delta S_\lambda(k).$$

Proof. In this proof we ignore the fact that some of the a_i could be identical, and consider the d^k possible values (not necessarily distinct) of the product $X_1 X_2 \cdots X_k$. Let the set G be

$$G = \left\{ (i_1, i_2, \ldots, i_k) : \left(\frac{a_{i_1}}{d}\right)\left(\frac{a_{i_2}}{d}\right) \cdots \left(\frac{a_{i_m}}{d}\right) < \frac{1}{\lambda}, \qquad \text{some } m \leqslant k \right\}.$$

By the definition of $U(\lambda)$, we then have

$$P(U(\lambda) \leqslant k) = d^{-k}|G|.$$

We now look at the first m elements of the trajectory of n. By the mixing property of $C(n)$, each of the d^m orderings for the first m elements of the trajectory happens for a unique residue class (mod d^m). In other words, for any sequence $(i_1, i_2, \ldots, i_m)$, there is a unique $n^* \bmod d^m$ so that

$$C^{(m)}(n^*) = h_{i_1}\Big(h_{i_2}\big(\cdots (h_{i_m}(n^*)) \cdots\big)\Big). \tag{4}$$

By the form of $h_i(n)$, we can rewrite this as

$$C^{(m)}(n^*) = \left(\frac{a_{i_1}}{d}\right)\left(\frac{a_{i_2}}{d}\right) \cdots \left(\frac{a_{i_m}}{d}\right) n^* + c(i_1, i_2, \ldots, i_m),$$

with $c(i_1, i_2, \ldots, i_m)$ not dependent on n. This equation shows that $C^{(m)}(n^*) < n^*/\lambda$ is either true for all large members of the same residue class (mod d^m), or is false for all large members of the residue class.

By using the association of residue classes (mod d^k) with k-tuples $(i_1, i_2, \ldots, i_k)$ given by equation (4), we see that a residue class (mod d^k) has all large members in the set $S_\lambda(k)$ if and only if the k-tuple belongs to G. In other words,

$$\delta S_\lambda(k) = d^{-k}|G|,$$

which proves Theorem 1.

By Theorem 1, the study of $\sigma_\lambda(n)$ boils down to a study of the random variable $U(\lambda)$. For our purposes, the most important properties of $U(\lambda)$ are the following.

THEOREM 2. *If $a_0 a_1 \cdots a_{d-1} < d^d$, then*

(i) *for any $\lambda > 0$,*

$$\lim_{k \to \infty} P(U(\lambda) \leqslant k) = 1,$$

(ii) *if $c = d/\ln(d^d/a_0 a_1 \cdots a_{d-1})$, then*

$$\lim_{\lambda \to \infty} E(U(\lambda))/\ln \lambda = c, \textit{ and}$$

(iii) $\lim_{\lambda \to \infty} Var(U(\lambda))/\ln \lambda = 0$,
(*where E and Var are the expected value and the variance*, *respectively*).

(Think of E as the mean, and Var as the square of the standard deviation.)

Proof. By the definition of X_i,

$$X_1 X_2 \cdots X_n = (a_0/d)^{k_0} (a_1/d)^{k_1} \cdots (a_{d-1}/d)^{k_{d-1}},$$

for some $k_0 + k_1 + \cdots + k_{d-1} = n$. By Borel's Law of Large Numbers ([**13**], p. 304),

$$P\left(\lim_{n \to \infty} \frac{k_i}{n} = \frac{1}{d} \right) = 1$$

or, more simply stated, each residue class gets its fair share as $n \to \infty$. By writing $k_i = (n/d)(1 + \varepsilon_i)$, we have

$$X_1 X_2 \cdots X_n = \left(\frac{a_0}{d} \right)^{(1+\varepsilon_0)} \left(\frac{a_1}{d} \right)^{(1+\varepsilon_1)} \cdots \left(\frac{a_{d-1}}{d} \right)^{(1+\varepsilon_{d-1})},$$

and we know that each $\varepsilon_i \to 0$ as $n \to \infty$, with probability 1. Our condition on the a_i is

$$a_0 a_1 \cdots a_{d-1}/d^d < 1,$$

so as $n \to \infty$, $X_1 X_2 \cdots X_n \to 0$, with probability 1. Therefore, $U(\lambda) < \infty$ with probability 1, for any $\lambda > 0$.

This also shows that

$$P\left(\lim_{n \to \infty} \frac{X_1 X_2 \cdots X_n}{\left(a_0 a_1 \cdots a_{d-1}/d^d \right)^{n/d}} = 1 \right) = 1,$$

or that all the action becomes concentrated where we expect it. Consequently, by the definition of $U(\lambda)$, for large λ we have

$$\left(a_0 a_1 \cdots a_{d-1}/d^d \right)^{U(\lambda)/d} \sim \frac{1}{\lambda},$$

or $U(\lambda) \sim c \ln \lambda$ where c is defined in the statement of the theorem. This is the same as part (ii). Finally, since all the action becomes concentrated, $Var(U(\lambda))/\ln \lambda \to 0$ as $\lambda \to \infty$.

As an immediate result of Theorem 1 and Theorem 2(i), we have the main result, namely

COROLLARY. *If $C(n)$ is a mixing Collatz-type iteration problem*, *with $a_0 a_1 \cdots a_{d-1} < d^d$*, *then for any $\lambda > 0$*,

$$\lim_{k \to \infty} \delta S_\lambda(k) = 1,$$

or, *in words*, *almost every integer n has the property that some iterate $C^{(k)}(n)$ is less than n/λ.*

Before we present our conjecture and numerical calculations, we note that the Corollary is, in some sense, the best possible. That is, if $C(n)$ is mixing with $a_0 a_1 \cdots a_{d-1} > d^d$, then $\lim_{k \to \infty} \delta S_\lambda(k) \neq 1$.

It must be stressed here that these probabilistic results deal only with $\sigma_\lambda(n)$, and not with $\sigma_\infty(n)$. Consequently, the original Collatz-type problem is far from answered. The probabilistic analysis does give us insight into the problem, however.

Theorem 1 shows that the first few iterates (with respect to n) of $C(n)$ behave like the probabilistic model. We suspect that the iterates of $C(n)$ continue to behave this way until a cycle is reached. Since $\sigma_\infty(n)$ is the number of iterations of $C(n)$ needed to reach r, for some $r \in R$, we can consider it to be counting the number of steps it takes to decrease by a factor of r/n. Therefore, we expect $\sigma_\infty(n)$ to behave like $U(n/r)$ for large n. Further, since by Theorem 2, $U(n/r)$ behaves like $c \ln(n/r) \sim c \ln n$, we make the following conjecture. (Note that the standard deviation is the square root of the variance.)

CONJECTURE. *If $C(n)$ is mixing, and $a_1 a_2 \cdots a_{d-1} < d^d$, then $\sigma_\infty(n) < \infty$ for all n; and further, as $N \to \infty$, the set of points $\{\sigma_\infty(n)/\ln n\colon\ n \leqslant N\}$ has mean approaching $c = d/\ln(d^d/a_0 a_1 \cdots a_{d-1})$, and standard deviation approaching* 0.

In our computer analysis we do not deal with the function $\sigma_\infty(n)$ directly. We define a new function $\sigma^K(n)$ to be the first k so that $C^{(k)}(n) \leqslant K$, where K is a fixed constant. Once we have verified $\sigma_\infty(n) < \infty$ for all $n < L$, the conjecture is obviously equivalent to the following.

CONJECTURE*. *For fixed K and L, $K < L$, the set $\{\sigma^K(n)/\ln(n/K)\colon\ L \leqslant n \leqslant N^*\}$ has mean approaching c and standard deviation approaching* 0 *as N^* goes to infinity.*

Testing this new statement is much easier, since by choosing K and L we can eliminate erratic behavior of small n and the worry about different cycles. In our computer testing, we used $K = 100$ and $L = 1001$, for the $3x+1$ problem, the $3x-1$ problem, and the $4x \pm 1$ problem. TABLE 1 gives the values for various N^*, along with the conjectured values at $N^* = \infty$.

	Problem					
	$3x+1$		$3x-1$		$4x \pm 1$	
N^*	mean	s.d.	mean	s.d.	mean	s.d.
5,000	7.291	5.070	6.969	4.640	6.687	5.375
10,000	7.232	4.722	6.985	4.433	6.514	4.901
20,000	7.145	4.372	6.992	4.223	6.401	4.478
50,000	7.052	3.972	7.003	4.006	6.281	4.058
100,000	7.018	3.741	7.007	3.830	6.211	3.787
1,000,000	6.963	3.206	7.008	3.348	6.070	3.172
10,000,000	—	—	—	—	5.992	2.777
⋮						
∞(?)	6.952	0.0	6.952	0.0	5.733	0.0

TABLE 1

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PROBLEMS

LEROY F. MEYERS, Editor
G. A. EDGAR, Associate Editor
The Ohio State University

Proposals

To be considered for publication, solutions should be mailed before October 1, 1985.

***1216.** Find all differentiable functions f that satisfy

$$f(x) = xf'\left(\frac{x}{\sqrt{3}}\right) \quad \text{for all real } x.$$

[*Stanley Rabinowitz, Merrimack, New Hampshire.*]

1217. A die is rolled repeatedly. Let p_n be the probability that the accumulated score is at some time equal to n. Find $\lim_{n\to\infty} p_n$. [*David Callan, Lafayette College.*]

1218. Begin with a list of n 1's. Adjoin the sum of the first two numbers in the list to the end of the list. Then adjoin the sum of the third and fourth numbers to the end of the list. Continue adjoining sums of pairs to the end of the list until no pair remains to be summed.

(a) How long is the final list?
(b) What is its last entry?
(c) What is the sum of the numbers in the final list?

[*Vic Norton, Bowling Green State University.*]

1219. Sum the infinite series

$$\sum_{k=2}^{\infty}\left(\frac{(-1)^k}{k+1}(\zeta(k)-1)\right) \quad \text{and} \quad \sum_{k=2}^{\infty}\left(\frac{1}{k+1}(\zeta(k)-1)\right),$$

where $\zeta(k) = \sum_{n=1}^{\infty} n^{-k}$ (the Riemann zeta function). [*L. Matthew Christophe, Jr., Wilmington, Delaware.*]

1220. Prove that if the odd prime p divides $a^b - 1$, where a and b are positive integers, then p appears to the same power in the prime factorization of $b(a^d - 1)$, where d is the greatest common divisor of b and $p - 1$. [*Gregg Patruno, student, Princeton University.*]

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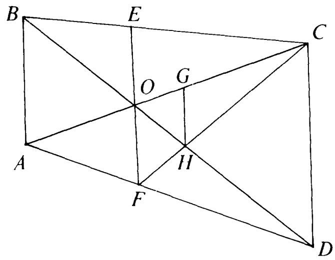
*We invite readers to submit problems believed to be new. Proposals should be accompanied by solutions, if at all possible, and by any other information that will assist the editors. A problem submitted as a Quickie should have an unexpected, succinct solution. An asterisk (*) will be placed next to a problem number to indicate that the proposer did not supply a solution.*

Solutions should be written in a style appropriate for Mathematics Magazine. Each solution should begin on a separate sheet containing the solver's name and full address. It is not necessary to submit duplicate copies.

Send all communications to the problems department to Leroy F. Meyers, Mathematics Department, The Ohio State University, 231 W. 18th Ave., Columbus, Ohio 43210.

Quickie

Answer to the Quickie is on p. 183.



Q697. In the figure, $ABCD$ is a trapezoid, AB, CD, EF, and GH are parallel, and the lines AC, BD, and EF are concurrent at O. Prove that E, G, and D are collinear. [*William A. McWorter, Jr., The Ohio State University.*]

Solutions

All Digits Even **March 1984**

1189. $.8^2 + .6^2 = 1$. How many other pairs of positive real numbers x, y are there whose decimal expansions contain only even digits and satisfy $x^2 + y^2 = 1$? [*James Propp, student, Cambridge University.*]

Solution (adapted by the editor): Let E_n be the set of numbers in $[0,1)$ whose first n digits (after the decimal point) are even, and let $E_\infty = \bigcap_{n=1}^{\infty} E_n$. We will show that there exist uncountably many pairs of numbers x, y in E_∞ such that $x^2 + y^2 = 1$. It is easy to check that for all such pairs with $x \geqq y$ we have either $x = .8$ (and $y = .6$) or $x \geqq .88$.

For $0 \leqq t \leqq 1$ let $f(t) = \sqrt{1 - t^2}$, and for $n \geqq 2$ let $D_n = \{x \in E_n\colon f(x) \in E_{n-1}\}$. We construct subsets F_n of D_n for $n \geqq 2$ by induction. Let $F_2 = \{.88\}$. (Note that $f(.88) = .47\ldots \in E_1$.) If F_{n-1} has been constructed, let

$$F_n = \{x + 2j \cdot 10^{-n} \in D_n\colon x \in F_{n-1} \quad \text{and} \quad j \in \{0,1,2,3,4\}\}.$$

Each element of F_n will be an n-place terminating decimal. We call $x_j = x + 2j \cdot 10^{-n} \in F_n$ a *child* of $x \in F_{n-1}$. (Note that if $j = 0$, then x_0, if an element of F_n, is to be distinguished from x as an element of F_{n-1}.) The children of .88 in F_2 are .884 and .886 in F_3; the children of .884 are .8844 and .8846, and the children of .886 are .8864, .8866, and .8868 in F_4. An element z of F_n is *bad* if the nth digit of $f(z)$ is 0; otherwise z is *good*. Our goal is to show that for $n \geqq 3$, every good element of F_{n-1} has at least two good children in F_n. A few useful results are stated first.

(A) If $.884 = .884\bar{0} \leqq t \leqq .\bar{8} = 8/9$, then $-5/2 < -1.95 < f'(t) < -1.89 < -5/3$. (*Proof.* f' is monotone.)

(B) If z and z' are in $[.884, 8/9]$ and $z' - z = 2k \cdot 10^{-n}$ with $k = 1,2,3,4$, then $10^n(f(z) - f(z'))$ belongs to $(3.78, 3.9)$, $(7.56, 7.8)$, $(11.34, 11.7)$, $(15.12, 15.6)$, respectively. (*Proof.* The mean value theorem and the strong bounds in (A).)

(C) If $r_0, \ldots, r_4$ are any real numbers such that $-1/2 < r_j - r_{j-1} < -1/3$ for $j = 1,2,3,4$, then two or three of the r_j's have even integral part. (*Proof.* The inequalities imply that two or three of the r_j's have integral part $[r_0] - 1$, the others having integral part $[r_0]$ or $[r_0] - 2$.)

Suppose now that x is a good element of F_{n-1}. Let $x_j = x + 2j \cdot 10^{-n}$ and $r_j = 10^{n-1} f(x_j)$ for $j = 0,1,2,3,4$. By (A) and the mean value theorem we have $r_j - r_{j-1} > 10^{n-1}(-5/2) \cdot 2 \cdot 10^{-n} = -\frac{1}{2}$, and, similarly, $r_j - r_{j-1} < -\frac{1}{3}$. Hence by (C), two or three of the $[r_j]$'s are even, and so the $(n-1)$st digits of these $f(x_j)$'s are even. The corresponding x_j's are thus in F_n, since the first

$n-2$ digits of $f(x_j)$ are the same as those of $f(x)$. (This is obvious if the $(n-1)$st digit of $f(x)$ is at least 2; and is true also if the $(n-1)$st digit of $f(x)$ is 1, since in that case the $(n-1)$st digit of $f(x_j)$, which must be even, is 0.) Furthermore, if some $x_j \in F_n$ is bad, i.e., the nth digit of $f(x_j)$ is 0, then x_{j-1} and x_{j-2} (subscripts taken modulo 5) are good children of x. For example, if x_1 is a bad child (the hardest case), then the units digit of $[10r_1]$ is 0, so that, from (B), the units digit of $[10r_0]$ is 3 or 4, the $n-1$ preceding digits being the same as those of $[10r_1]$. Hence x_0 is a good child of x. From (B) again we see that the units digit of $[10r_4]$ is 8 or 9, the tens digit being 2 less than that of $[10r_1]$ and $[10r_0]$. (Note that the tens digit of $[10r_0]$, or $[10^n f(x)]$, is even since $x_0 \in F_n$, and is not 0 since x is good. Hence the first $n-2$ digits of $f(x_4)$ are the same as those of $f(x)$, and the next digit is even.) Hence x_4 is a good child of x.

Since every good element of F_{n-1} has two or three good children, the resulting binary-ternary tree contains uncountably many paths, each leading to a distinct infinite decimal $x \in \bigcap_{n=1}^{\infty} E_n$ with corresponding $y = f(x) \in \bigcap_{n=2}^{\infty} E_{n-1}$.

A loose probabilistic "argument" for the result just proved runs as follows. There are 5^n n-digit terminating decimal fractions x in E_n. For each of these, the odds are 1 out of 2^n that $\sqrt{1-x^2}$ has its first n digits even. Hence we expect to find on the order of $(5/2)^n$ pairs. When n becomes sufficiently large, we expect the "law of averages" to take over: if our x's have not become extinct, they should proliferate exponentially.

JAMES G. PROPP, student
University of California, Berkeley

Two incorrect solutions were received.

The smallest x in F_{25} is given by (thanks to G. A. Edgar and his microcomputer)

$$x = .8844220066042020020020604, \qquad f(x) = .4666880266668484286088627\ldots.$$

Two related, unanswered questions may be raised. Can x (assumed greater than .84) be a terminating decimal in E_∞ such that $f(x)$ is also a terminating decimal in E_∞? (There is no such x with up to 36 digits.) More generally, can x and $f(x)$ be rational? (There is no such x with denominator up to 40000.)

Mean Area of Inscribed Triangles — May 1984

1191. Let $P_1, P_2, \ldots, P_n$ be the vertices of a regular polygon inscribed in the unit circle. Let A_n denote the sum, and B_n the arithmetic mean, of the areas of all triangles $P_1P_jP_k$ for $1 < j < k \leqslant n$. Evaluate A_n, show that A_n decreases as $n \to \infty$, and evaluate $\lim_{n\to\infty} B_n$. [*L. Kuipers, Sierre, Switzerland.*]

Editor's error: "A_n decreases" should be "A_n increases" or "B_n decreases."

Composite solution: Place the circle in the complex plane so that the center is at the origin and the affix of P_m is $\cos((m-1)\theta) + i\sin((m-1)\theta)$ for $m = 1, 2, \ldots, n$, where $\theta = 2\pi/n$. Then for $1 < j < k \leqq n$ the area of triangle $P_1P_jP_k$ is

$$\frac{1}{2}\begin{vmatrix} 1 & 0 & 1 \\ \cos((j-1)\theta) & \sin((j-1)\theta) & 1 \\ \cos((k-1)\theta) & \sin((k-1)\theta) & 1 \end{vmatrix}$$

$$= \frac{1}{2}[\sin((j-1)\theta) + \sin((k-j)\theta) - \sin((k-1)\theta)]$$

$$= \frac{1}{2}[\sin((j-1)\theta) + \sin((k-j)\theta) + \sin((n-k+1)\theta)].$$

Among all allowable pairs (j,k), each of the numbers $j-1$, $k-j$, and $n-k+1$ assumes the value p exactly $n-p-1$ times, if $1 \leqq p \leqq n-1$. Hence

$$A_n = \frac{3}{2}\sum_{p=1}^{n-1}(n-p-1)\sin(p\theta) = \frac{3}{2}\left[(n-1)\sum_{p=0}^{n-1}\sin(p\theta) - \sum_{p=0}^{n-1}p\sin(p\theta)\right].$$

But the sums are just the imaginary parts of

$$\sum_{p=0}^{n-1}\omega^p = \frac{1-\omega^n}{1-\omega} = 0 \quad\text{and}\quad \sum_{p=0}^{n-1}p\omega^p = \frac{1}{1-\omega}\left[\sum_{p=1}^{n}\omega^p - n\omega^n\right] = -\frac{n}{1-\omega},$$

where $\omega = \exp(2\pi i/n)$, so that

$$\sum_{p=0}^{n-1}\sin(p\theta) = 0 \quad\text{and}\quad \sum_{p=0}^{n-1}p\sin(p\theta) = \operatorname{Im}\left(-\frac{n}{1-\cos\theta - i\sin\theta}\right) = -\frac{n}{2}\cot\frac{\pi}{n}.$$

Hence

$$A_n = \frac{3n}{4}\cot\frac{\pi}{n},$$

which obviously increases to ∞ as $n \to \infty$.

Now

$$B_n = \frac{A_n}{\binom{n-1}{2}} = \frac{3n}{2(n-1)(n-2)}\cot\frac{\pi}{n},$$

so that

$$\lim_{n\to\infty} B_n = \lim_{n\to\infty}\frac{3n^2}{2\pi(n-1)(n-2)}\cdot\lim_{n\to\infty}\frac{\pi/n}{\tan(\pi/n)} = \frac{3}{2\pi}.$$

Lastly, we show that B_n decreases as $n \to \infty$. Since $B_3 = \frac{3}{4}\sqrt{3} > 1 = B_4$, we need do this analytically only for $n \geqq 4$. If we set

$$f(x) = \frac{3x}{2(x-1)(x-2)}\cot\frac{\pi}{x} \quad\text{for}\quad x > 2,$$

then showing that $f'(x) < 0$ is equivalent to showing that

$$\frac{x}{2\pi}\sin\frac{2\pi}{x} > \frac{(x-1)(x-2)}{x^2-2}.$$

But if $x \geqq 2$, then $0 < 2\pi/x \leqq \pi$ and

$$\frac{x}{2\pi}\sin\frac{2\pi}{x} > 1 - \frac{1}{6}\left(\frac{2\pi}{x}\right)^2;$$

and if $x \geqq 4$, then $9x^3 + 4\pi^2 > 36x^2 > (2\pi^2+12)x^2$, so that

$$\frac{1}{6}\left(\frac{2\pi}{x}\right)^2 < \frac{3x-4}{x^2-2}.$$

Thus

$$\frac{x}{2\pi}\sin\frac{2\pi}{x} > 1 - \frac{1}{6}\left(\frac{2\pi}{x}\right)^2 > 1 - \frac{3x-4}{x^2-2} = \frac{(x-1)(x-2)}{x^2-2},$$

and B_n decreases to $3/2\pi$, as claimed.

DANIEL ROPP, student
Stillman Valley High School
Illinois

and

ROGER B. NELSEN
Lewis and Clark College,
independently

Also solved by Duane M. Broline, David Graves, Hans Kappus (Switzerland), J. C. Linders (The Netherlands), Richard Parris, Paul J. Zwier, and the proposer. Neither the solvers nor the editor (until now) noticed that this problem, except for proof of monotonicity, is equivalent to problem 1104 (proposed v. 53 (1980) 244, solved v. 54 (1981) 273).

Matrices with Zero Trace **May 1984**

1192. Let M be an $n \times n$ matrix with complex elements such that $\operatorname{tr} M = 0$. Prove:

(i) $x^*Mx = 0$ for some nonzero vector x;

(ii) there are unitary matrices U such that U^*MU has all its diagonal elements 0. (M^* denotes the conjugate transpose of M.) [*The late H. Kestelman, University College London, England.*]

Comment: A solution may be found in the paper "On similarity and the diagonal of a matrix," by P. A. Fillmore, *American Mathematical Monthly*, v. 76 (1969), pp. 167–169, Corollary 1.

A. R. SOUROUR
University of Victoria

Also solved, generalized, or commented on by Wayne Barrett, Enzo R. Gentile (Argentina), Thomas Lowell Markham, John J. Martinez, Dennis Spellman, and the proposer.

Several solvers quoted the result of F. Hausdorff, "Der Wertvorrat einer Bilinearform," *Mathematische Zeitschrift*, v. 3 (1919), pp. 314–316, that $\{x^*Mx\colon\ x^*x = 1 \text{ and } x \in \mathbf{C}^n\}$, the field of values of the matrix M, is convex.

A Sum of Products of Binomial Coefficients **May 1984**

1193. Let $a_1, a_2, \ldots, a_m$ be fixed nonnegative integers, and let n be a fixed integer with $n \geqslant \sum_{i=1}^m a_i$. Show that

$$\sum_* \prod_{i=1}^m \binom{k_i}{a_i} = \binom{n+m-1}{n - \sum_{i=1}^m a_i},$$

where the *-sum is over all ordered m-tuples $(k_1, k_2, \ldots, k_m)$ of integers that sum to n. [*Allen J. Schwenk, U.S. Naval Academy.*]

Solution I: Count the arrangements into m rows of $s = a_1 + \cdots + a_m$ white balls and $n - s$ blue balls, such that row i contains exactly a_i white balls for $i = 1, 2, \ldots, m$. The left side of the identity to be proved results from partitioning such arrangements according to the total number, k_i, of balls in row i. The right side arises from the observation that the a_i white balls in row i create $a_i + 1$ "slots" in row i, hence $\sum_{i=1}^m (a_i + 1) = s + m$ "slots" altogether, in each of which some or none of the $n - s$ blue balls may be inserted. The number of ways of so distributing the blue balls is just the number of ways of expressing $n - s$ as an ordered sum of $s + m$ nonnegative integers, namely (Berman and Fryer, *Introduction to Combinatorics*, §4.1, Th. 2)

$$\binom{(n-s)+(s+m)-1}{n-s} = \binom{n+m-1}{n - \sum_{i=1}^m a_i}.$$

CARL WAGNER
University of Tennessee

Solution II: The coefficient of x^p in the power series development of $(1-x)^{-q}$ is $(-1)^p\binom{-q}{p} = \binom{p+q-1}{p}$. Then in the expansion of $(1-x)^{-(a_i+1)}$, the coefficient of $x^{k_i-a_i}$ is $\binom{k_i}{k_i-a_i} = \binom{k_i}{a_i}$, where $k_i = a_i, a_i+1, \ldots$. For fixed k_i, where $i = 1, \ldots, m$, the product $\binom{k_1}{a_1}\binom{k_2}{a_2}\cdots\binom{k_m}{a_m}$ will be the contribution to the coefficient of $x^{\Sigma_{i=1}^m (k_i - a_i)}$ due to the choice of the k_i's in the development of

$$P = (1-x)^{-(a_1+1)}(1-x)^{-(a_2+1)}\cdots(1-x)^{-(a_m+1)}.$$

So $\Sigma_* \Pi_{i=1}^m \binom{k_i}{a_i}$ is the total coefficient of x^{n-s}, where $n = \Sigma_{i=1}^m k_i$ and $s = \Sigma_{i=1}^m a_i$. But $P = (1-x)^{-(s+m)}$; hence the coefficient of x^{n-s} is $\binom{n+m-1}{n-s}$.

GRAHAM LORD
Princeton, New Jersey

Also solved as in Solution I by Duane M. Broline, James Propp, and the proposer; as in Solution II by Jens Schwaiger (Austria) and the proposer; and using induction by J. S. Frame, L. Kuipers (Switzerland), Robert E. Schafer, and Paul J. Zwier.

Solvable Differential Equations with Nonextendible Solutions **May 1984**

1194. Find a simple smooth function f of the two variables x and y such that:

(1) the differential equation $y' = f(x,y)$ is solvable in closed form, and

(2) the domain of definition of f is connected but not simply connected (and cannot be made so by continuous extension of f). [*J. Walter, RWTH Aachen, West Germany.*]

Solution: We seek a function f of the form

$$f(x,y) = \frac{g(x,y)}{x^2+y^2}.$$

Writing $dy/dx = f(x,y)$ as $g(x,y)dx - (x^2+y^2)dy = 0$, we see that the differential equation will be exact provided that $g_y(x,y) = -2x$. Hence we take $g(x,y) = -2xy$. Solving

$$\frac{dy}{dx} = -\frac{2xy}{x^2+y^2}$$

yields the closed form solutions

$$x^2y + \frac{1}{3}y^3 = c.$$

Now f is a smooth function defined at all points of the plane except the origin. Letting $x = r\cos\theta$ and $y = r\sin\theta$ gives $f(x,y) = -\sin(2\theta)$, which shows that $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist. Hence f cannot be extended continuously from its non-simply connected domain of definition, the punctured plane, to the entire simply connected plane.

PETER W. LINDSTROM
St. Anselm College

Also solved by Richard Parris, Michael P. Tangredi, and the proposer.

All solvers expressed $f(x,y)$ as $g(x,y)/h(x,y)$, where g and h are continuously differentiable in some neighborhood of the origin, and h is positive-valued in the neighborhood, except that $h(0,0) = 0$. Lindstrom, Parris, and Tangredi used $h(x,y) = x^2 + y^2$ and solved for g, whereas the proposer used $g(x,y) = 1$ and $h(x,y) = 3x^2 + 4(y-x^3)^2/\left(1+\sqrt{1+4x(y-x^3)}\right)^2$. If h is given, then we may let $g(x,y) = \psi(x) - \int_0^y h_x(x,y)\,\partial y$, where ψ is any continuously differentiable function.

1195. Phone books, n in number, are kept in a stack. The probability that the book numbered i (where $1 \leqslant i \leqslant n$) is consulted for a given phone call is $p_i > 0$, where $\Sigma_{i=1}^{n} p_i = 1$. After a book is used, it is placed at the top of the stack. Assume that the calls are independent and evenly spaced, and that the system has been employed indefinitely far into the past. Let d_i be the average depth of book i in the stack. Show that $d_i \leqslant d_j$ whenever $p_i \leqslant p_j$. Thus, on the average, the more popular books have a tendency to be closer to the top of the stack. (Compare problem 1159, vol. 57 (1984), p. 50). [*Boris Pittel, The Ohio State University.*]

Editor's error: "$p_i \leqq p_j$" should be "$p_i \geqq p_j$."

Solution: Let $p_{ij} = \Pr\{\text{book } i \text{ is above book } j\}$. Then the average depth of book j is

$$d_j = \sum_{i \neq j} p_{ij},$$

where the top book is considered to be at depth 0. Now if book i is above book j, then the relative order of books i and j is changed after a call if and only if book j is consulted. Hence,

$$p_{ij} = p_{ij}(1 - p_j) + p_{ji} p_i = p_{ij}(1 - p_j) + (1 - p_{ij}) p_i = p_i + p_{ij}(1 - p_i - p_j).$$

Thus we have

$$p_{ij} = \frac{p_i}{p_i + p_j}$$

and

$$d_j = \sum_{k \neq j} \frac{p_k}{p_k + p_j}.$$

If $p_i \geqq p_j$, then $p_k/(p_k + p_i) \leqq p_k/(p_k + p_j)$ for $k \notin \{i, j\}$, and $p_j/(p_j + p_i) \leqq p_i/(p_i + p_j)$. Since each term in the sum for d_i is less than or equal to the corresponding term in the sum for d_j, we have $d_i \leqq d_j$.

VÍCTOR HERNÁNDEZ
Universidad Autónoma de Madrid, Spain

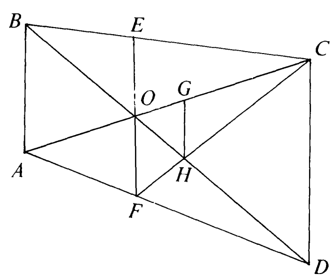
Also solved by James Propp (student) and the proposer.

Man-Keung Siu (Hong Kong) notes that the result may be found in K. Lam, M.-Y. Leung, and M.-K. Siu, "Self-organizing files," *UMAP Journal*, v. 4 (1983) pp. 51–84 (*UMAP* unit 612). This problem is referred to in the literature as the "move-to-front" scheme, which in the long run is less efficient than the "transposition" scheme. See also W. J. Hendricks, "The stationary distribution of an interesting Markov chain," *J. Applied Probability*, v. 9 (1972), pp. 231–233.

Answer

Solution to the Quickie on p. 178.

Q697. The figure is the perspective projection of a rectangle whose side AB is parallel to the image plane. The original figure (with EG and GD drawn) is symmetric with respect to the perpendicular bisector of AB.



REVIEWS

PAUL J. CAMPBELL, Editor
Beloit College

Assistant Editor: Eric S. Rosenthal, West Orange, NJ. Articles and books are selected for this section to call attention to interesting mathematical exposition that occurs outside the mainstream of the mathematics literature. Readers are invited to suggest items for review to the editors.

Broder, Andrei, and Stolfi, Jorge, *Pessimal algorithms and simplexity analysis*, SIGACT News 16:3 (Fall 1984) 49-53.

"Intuitively, a reluctant algorithm for a problem P is one which wastes time in a way that is sufficiently contrived to fool a naive observer. We can make this concept mathematically precise... ." And they do, in this venerable founding paper on reluctant algorithmics, which includes the memorable new algorithms backwards-first search and slowsort (the latter being "the eminently suitable algorithm whenever your boss sends you to sort something in Paris").

Denning, Peter J., *The science of computing: what is computer science?*, American Scientist 73 (January-February 1985) 16-19.

First installment of a new column devoted to computer science. Denning cites 11 areas in the field and outlines their content, fundamental questions, and major accomplishments. He cites as the fundamental question underlying all of computer science: What can be automated?

Guillen, Michael, *Karmarkar's algorithm*, Science Digest 93:3 (March 1985) 24.

Reveals some new facts about the new algorithm for linear programming. Karmarkar estimates that if the simplex method requires n iterations, his method will require only log n. Ron Graham (Bell Labs) notes that AT&T will attempt to protect both the practical procedure (which has not yet been divulged) and the theoretical mathematical concept underlying it: "It remains to be decided in the courts whether mathematical theorems can be copyrighted or patented."

McEliece, Robert J., *The reliability of computer memories*, Scientific American 252:1 (January 1985) 88-95, 120.

Describes how error-correcting codes help computer memories recover from alpha particle attack, potentially extending a one-megabyte memory's mean time before failure from days to years.

Lavenda, Bernard H., *Brownian motion*, Scientific American 252:2 (February 1985) 70-85, 128.

Popular culture identifies Einstein with relativity. Much less well-known is his 1905 theory of Brownian motion. It not only verified the physical existence of the atom (and showed how to measure its mass accurately) but also established statistical mechanics as the basis of thermodynamics. This article captures the drama and importance of his discovery and subsequent developments.

Winkler, Karen J., *A leading computer scientist bemoans our "love affair" with the machine*, Chronicle of Higher Education (6 February 1985) 35-36.

Joseph Weizenbaum (MIT) warns that scholarship may be determined by what data is available in machine-readable form, all other data being neglected. The computer may also modify the language in which the researcher thinks; Weizenbaum stresses our need to be more conscious of the impacts of the computer.

McKean, Kevin, and Dworetzky, Tom, *Fuzzy means to logical ends*, Discover 6:2 (February 1985) 70-73.

Fuzzy logic, an outgrowth of fuzzy set theory (in which set membership is a matter of degree), is proving popular in artificial intelligence research. The attraction is that it can accommodate imprecise categories or observations; of course, it delivers a correspondingly imprecise conclusion. Critics worry that fuzzy logic will be used as a respectable cover for fuzzy thinking.

Looking for a busy beaver, Science News 127 (February 1985) 89.

How quickly does complexity develop in small machines? A Turing machine with three states can write at most six ones before halting; with four states, it can do 13 ones. Now George Uhing, an amateur mathematician, has found a five-state Turing machine that prints 1915 ones as it goes through more than two million moves before halting. Is this the maximum? No one knows; there are 6.4×10^{13} different five-state Turing machines to try.

Hargittai, István, and Lengyel, Györgyi, *The seven one-dimensional space-group symmetries illustrated by Hungarian folk needlework*, J. Chemical Education 61:12 (December 1984) 1033-1034; *The seventeen two-dimensional space-group symmetries in Hungarian needlework*, ibid. 62:1 (January 1985) 35-36.

Hungarian table covers, pillow slips, bed sheet edgings, shirt fronts, children's bags, and peasant vests are the canvases upon which one can observe embroidery in all the plane repeating patterns (not counting any color symmetries or black-white reversals).

Kennedy, Dan, *Mathematics in the real world, really*, Mathematics Teacher (January 1985) 18-22.

Entertaining and inspiring account of a few of the author's "magic moments," when his mathematical training serendipitously supplied him with the key to solving a problem in the Real World.

Taubes, Gary, *The mathematics of chaos*, Discover 5:9 (September 1984) 30-43.

Henri Poincaré was dismayed to discover at the turn of the century a deterministic equation that could produce apparently random results. Subsequent research, from meteorology to medicine, has revealed that this phenomenon of chaos is not an anomaly but a common occurrence in our world.

Schechter, Bruce, *The maestro of the algorithm*, Discover 5:9 (September 1984) 74-78.

Personality portrait of Donald Knuth.

Freedman, Daniel Z., and van Nieuwenhuizen, Peter, *The hidden dimensions of spacetime*, Scientific American 252:3 (March 1985) 74-81, 128.

Spacetime, usually thought of as four-dimensional, must have seven or more additional dimensions in order to account for the four basic forces of nature.

Begley, Sharon, *Images of hyperspace*, Newsweek (17 December 1984) 87.

Impressions of the first world conference on 4-D, which attracted 700 fans.

Gardner, Martin, *Slicing pi into millions*, Discover 6:1 (January 1985) 50-52.

The master of recreational mathematics returns! In 1909 William James doubted that pi would ever be computed to 1000 places; actually, it was done to 1120 places in 1945 on a pre-electronic desk calculator. Researchers are now approaching the hundred millionth digit, with the billionth not inconceivable. Modern researchers use an algorithm discovered by Gauss; so far, no order is apparent in the digits.

Gardner, Martin, *666 and all that*, Discover 6:2 (February 1985) 34-35.

Numerology of the "number of the beast," by Dr. Matrix's close friend.

Gensler, Harry J., Godel's [sic] Theorem Simplified, University Pr of America, 1984; iii + 83 pp, $13.50, $7.25 (P).

Understandable proof of a version of Gödel's incompleteness theorem. Is it too much to ask of the author, however, that he spell the master's name correctly (either Gödel or Goedel) at least once?

Hollander, Myles, and Proschan, Frank, The Statistical Exorcist: Dispelling Statistics Anxiety, Dekker, 1984; xiii + 247 pp, $17.95.

Understandable and enjoyable presentation of how statistics is used in everyday life, via 26 vignettes. A novel feature is that the book uses no algebraic symbols. The authors assert that mass ignorance of statistics persists because books on it are written in a "foreign language": they "explain the mechanics of statistics using the language of algebra." A course based on this book would be a valuable addition to the high-school mathematics curriculum; it could also be taught in college for credit, just as credit is given for literature in translation, art and music appreciation, and non-lab science courses.

ApSimon, Hugh, Mathematical Byways in Ayling, Beeling, and Ceiling, Oxford U Pr, 1984; xii + 97 pp, $9.95.

An imaginative and entertaining collection of 11 original problems and their extensions. Set in the context of the imaginary English villages named in the title and the interactions of their idiosyncratic inhabitants, the problems can be solved with careful and ingenious application of high-school mathematics.

Dewdney, A. K., The Planiverse: Computer Contact with a Two-Dimensional World, Poseidon Pr, 1984; 267 pp, $9.95 (P).

E. A. Abbott's Flatland explored only a little of the possibilities of life-forms and technology in 2-D. Dewdney establishes contact through his computer with Arde, a different 2-D-world from Abbott's. Yendred, Dewdney's guide to Arde, pursues a geographic and spiritual odyssey that makes the experience more than a technical one, for both Dewdney and the reader.

Page, Warren (ed.), American Perspectives on the Fifth International Congress on Mathematical Education (ICME5), MAA, 1985; viii + 134 pp.

If you could not go to Adelaide last summer, here is a way to taste the flavor of the discussions at ICME5, courtesy of reports from the 29 U.S. participants who received NSF travel grants. Topics vary from curriculum issues to the place of computers, from contests and competitions to teacher shortages.

Bolter, J. David, Turing's Man: Western Culture in the Computer Age, U North Carolina Pr, 1984; xii + 264 pp, $19.95, $8.95 (P).

Like J. Weizenbaum's Computer Power and Human Reason, this book--written by a classicist with a master's in computer science--offers for humanists an explanation of computers and their functioning, and for technologists a recall to humanistic values in technology. "Turing's man" views humans as information processors, and nature as information to be processed; "he" is also "insensitive to the historical and intellectual context for his work." Yet Bolter has faith that reactive humanists will learn about computer technology and act to trim its excesses. (Surprisingly, Weizenbaum is not even mentioned.)

Cooke, Roger, The Mathematics of Sonya Kovalevskaya, Springer-Verlag, 1984; xiii + 234 pp, $29.80.

Despite the title, almost a third of the body of this book is (not unwelcome) biography. The remainder treats Kovalevskaya's work on PDE's, degenerate Abelian integrals, the shape of Saturn's rings, the Lamé equations, the Euler equations, and Bruns' theorem. A final chapter cites evaluations of her work, and appendices provide some mathematical background to some of the topics.

Lancy, David F., Cross-Cultural Studies in Cognition and Mathematics, Academic Pr, 1983; xxi + 248 pp.

Fascinating report of a four-year study, the Indigenous Mathematics Project, of the relationship between cultural background and pattern of cognitive development and acquisition of school arithmetic. Data were gathered in 12 highly diverse societies of Papua New Guinea. A key conclusion: "... universal schooling is creating an extremely dangerous situation... . [Cognitive] processes associated with schooling may be necessary for the production of airplane pilots, [but] they are unnecessary or even harmful for the production of subsistence horticulturists. How far would a child progress in mastering the village way of life if he firmly believed that answers are found in books, that problem-solving is an individual, intellectual activity, that effort is always and promptly rewarded, that the sexes are equivalent, and so forth?"

Wills, III, Herbert, Leonardo's Dessert: No Pi, NCTM, 1985; iv + 28 pp, $3.50 (P).

Recounts Leonardo da Vinci's experiments on scissors congruence of circular-arc curvilinear regions to rectilinear regions. The resulting quadratures avoid pi, because the area of the region is usually a rational multiple of the area of a square circumscribing the fundamental circle. The exposition is light, imaginative, and inspiring, even if the title will be considered slightly "distasteful" by some.

Burton, David M., The History of Mathematics: An Introduction, Allyn and Bacon, 1985; ix + 678 pp.

An intriguing and readable new history of mathematics, designed as a textbook (with problems) for an upper-level course for mathematics majors. As might be expected, however, the 19th and 20th centuries get short shrift; "topology" does not even appear in the index.

Giordano, Frank R., and Weir, Maurice D., A First Course in Mathematical Modeling, Brooks/Cole, 1985, xvii + 382 pp.

If mathematical modeling is the heart of applied mathematics, shouldn't we make it available at as early a point as possible, even to non-majors? This splendid text's only prerequisite is one-variable calculus. The book relies in part on the wealth of modeling material developed and formulated as UMAP modules. Unfortunately, the movement to incorporate early modeling experience into the curriculum has been preempted by the trend to incorporate discrete mathematics.

NEWS & LETTERS

GEOMETRIC DRAWING

As an engineer, I was entertained by the item "Geometry Strikes Again" in the January 1985 issue of *Mathematics Magazine*. Perhaps it would improve the standard of mathematical illustrations if students of mathematics, like their engineering counterparts, were required to take a course in engineering drawing, or at least that part which teaches the practical construction of projections. In making sketches of those dimensional shapes I still occasionally consult a text[1] which I used as a student thirty years ago. One source of trouble not mentioned in the item is the degradation of a once satisfactory illustration by repeated retracing from published material. I have seen some notable examples, but unfortunately haven't made a collection.

[1] W. Abbott, Practical Geometry and Engineering Graphics, Fifth edition, 1951, London: Blackie & Son.

-- L. P. Pook
Glasgow, Scotland

MAA "DISCOVERED"

Plane Embarassment was the article's title in Discover (April 1985, pp. 7-8) which summarized the story of the misdrawn MAA logo (this MAGAZINE, January 1985, pp. 12-28). Although the report savored the ironic situation, Discover's artist proved no better at rendering a correct icosahedron.

COMBINATORIAL BLOCK BUILDING

The usefulness of the block structure [described in "Child's Play", this MAGAZINE, March 1985, pp. 100-101] extends beyond suggesting solutions to combinatorial problems; it even includes teaching alternate derivations for combinatorial theorems. For instance, the formula for the number of combinations of n kinds of things taken r at a time,

$$\binom{n+r-1}{r} \qquad (*)$$

can be demonstrated by "stacking" unit "blocks" having n dimensions. In the rth "layer" of such a stacking, the position of each block represents a unique combination of moves in n positive directions -- the total of r moves which bring the block into its position from the origin. Thus the number of "blocks" in the rth "layer" is just the formula (*).

Other choice problems can be formulated and solved by counting "blocks" -- there seems to be a basic affinity between combinatorics and the block structure.

-- Warner Clements

MIAMI UNIVERSITY STATISTICS CONFERENCE

The Thirteenth Annual Mathematics and Statistics Conference at Miami University, Oxford, Ohio, will be held September 27 and 28, 1985 on the theme of "Statistics". Myles Hollander (Florida State Univ.), Richard L. Scheaffer (Univ. of Florida), and Ronald D. Snee (DuPont Corp.) are featured speakers. Contributed papers should be suitable for a diverse audience of statisticians, mathematicians, and students. Abstracts should be sent by June 1, 1985 to Professor Robert L. Schaefer, Dept. of Math. and Statistics, Miami University, Oxford, Ohio 45056. Information regarding preregistration and housing may also be obtained from Professor Schaefer.

Simultaneously, the Ohio Delta Chapter of Pi Mu Epsilon will hold its twelfth annual Student Conference. Students are invited to contribute papers, and should send abstracts by September 16, 1985 to Professor Milton Cox, at the above address.

1984 PUTNAM: SOLUTIONS

The following solutions to the 1984 W.L. Putnam competition problems were prepared for publication in this MAGAZINE by Loren Larson, St. Olaf College.

A-1. Let A be a solid $a \times b \times c$ rectangular brick in three dimensions, where $a, b, c > 0$. Let B be the set of all points which are at a distance of at most one from some point of A (in particular, B contains A). Express the volume of B as a polynomial in a, b, and c.

Sol. The set B can be partitioned and reassembled into the set consisting of A itself, six rectangular bricks of thickness 1, three cylinders of radius 1, and a sphere of radius 1, for a total volume of

$$abc + 2(ab+ac+bc) + \pi(a+b+c) + \frac{4}{3}\pi.$$

A-2. Express

$$\sum_{k=1}^{\infty} \frac{6^k}{(3^{k+1} - 2^{k+1})(3^k - 2^k)}$$

as a rational number.

Sol. Let $S(n)$ denote the nth partial sum of the given series. Then

$$S(n) = \sum_{k=1}^{\infty} \left[\frac{2^k}{3^k - 2^k} - \frac{2^{k+1}}{3^{k+1} - 2^{k+1}} \right]$$

$$= 2 - \frac{2^{n+1}}{3^{n+1} - 2^{n+1}},$$

and the series converges to

$$\lim_{n\to\infty} S(n) = 2.$$

A-3. Let n be a positive integer. Let a, b, x be real numbers, with $a \neq b$, and let M_n denote the $2n \times 2n$ matrix whose (i, j) entry m_{ij} is given by

$$m_{ij} = \begin{cases} x \text{ if } i = j, \\ a \text{ if } i \neq j \text{ and } i + j \text{ is even,} \\ b \text{ if } i \neq j \text{ and } i + j \text{ is odd.} \end{cases}$$

Thus, for example,

$$M_2 = \begin{pmatrix} x & b & a & b \\ b & x & b & a \\ a & b & x & b \\ b & a & b & x \end{pmatrix}.$$

Express $\lim_{x\to a} \dfrac{\det M_n}{(x - a)^{2n-2}}$ as a polynomial in a, b, and n, where $\det M_n$ denotes the determinant of M_n.

Sol. Subtract row 1 from rows $3,5,\ldots,2n-1$, and subtract row 2 from rows $4,5,\ldots,2n$. Then add columns $3,5,\ldots,2n-1$ to column 1, and add columns $4,6,\ldots,2n$ to column 2. This yields the blocked matrix

$$P_n = \begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix},$$

where $A_n = \begin{pmatrix} x+(n-1)a & nb \\ nb & x+(n-1)a \end{pmatrix}$,

C_n is a $(2n-2) \times 2$ zero matrix, and D_n is a $(2n-2) \times (2n-2)$ upper-triangular matrix with $a - x$ on the main diagonal. It follows that

$$\lim_{x\to a} \frac{\det M_n}{(x-a)^{2n-2}} = \lim_{x\to a} \frac{\det P_n}{(x-a)^{2n-2}}$$

$$= \lim_{x\to a} \frac{(\det A_n)(\det D_n)}{(x-a)^{2n-2}}$$

$$= \lim_{x\to a} \frac{[(x+(n-1)a)^2 - (nb)^2](x-a)^{2n-2}}{(x-a)^{2n-2}}$$

$$= n^2(a^2 - b^2).$$

A-4. A convex pentagon $P = ABCDE$, with vertices labeled consecutively, is inscribed in a circle of radius 1. Find the maximum area of P subject to the condition that the chords AC and BD be perpendicular.

Sol. Let $\theta = \text{Arc } AB$, $\alpha = \text{Arc } DE$, and $\beta = \text{Arc } EA$. Then $\text{Arc } CD = \pi - \theta$ and $\text{Arc } BC = \pi - \alpha - \beta$. The area of P, in terms of the five triangles from the center of the circle is

$$\frac{1}{2}\sin\theta + \frac{1}{2}\sin(\pi-\theta) + \frac{1}{2}\sin\alpha$$

$$+ \frac{1}{2}\sin\beta + \frac{1}{2}\sin(\pi-\alpha-\beta).$$

This is maximized when $\theta = \pi/2$ and $\alpha=\beta=\pi/3$. Thus, the maximum area is

$$\frac{1}{2}\cdot 1 + \frac{1}{2}\cdot 1 + \frac{1}{2}\cdot\frac{\sqrt{3}}{2} + \frac{1}{2}\cdot\frac{\sqrt{3}}{2} + \frac{1}{2}\cdot\frac{\sqrt{3}}{2}$$

$$= 1 + \frac{3}{4}\sqrt{3} .$$

A-5. Let R be the region consisting of all triples (x, y, z) of nonnegative real numbers satisfying $x + y + z \le 1$. Let $w = 1 - x - y - z$. Express the value of the triple integral

$$\iiint_R x^1 y^9 z^8 w^4 \, dx\, dy\, dz$$

in the form $a!b!c!d!/n!$, where a, b, c, d, and n are positive integers.

Sol. For $t > 0$, let R_t be the region consisting of all triples (x,y,z) of nonnegative real numbers satisfying $x + y + z \le t$. Let

$$I(t) = \iiint_{R_t} x^1 y^9 z^8 (t-x-y-z)^4 \, dx\, dy\, dz$$

and make the change of variables, $x = tu$, $y = tv$, $z = tw$. We see that $I(t) = I(1)t^{25}$. Let

$$J = \int_0^\infty I(t)e^{-t}dt. \text{ Then}$$

$$J = \int_0^\infty I(1)t^{25}e^{-t}dt = I(1)\Gamma(26) = I(1)25!.$$

It is also the case that J equals

$$\int_{t=0}^\infty \iiint_{R_t} e^{-t} x^1 y^9 z^8 (t-x-y-z)^4 dx\, dy\, dz\, dt .$$

Let $s = t - x - y - z$. Then $J =$

$$\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-s}e^{-x}e^{-y}e^{-z}\, x^1 y^9 z^8 s^4 dx\, dy\, dz\, ds$$

$$= \Gamma(2)\,\Gamma(10)\,\Gamma(9)\,\Gamma(5) = 1!9!8!4! .$$

The integral we desire is $I(1) = J/25! = 1!9!8!4!/25!$.

A-6. Let n be a positive integer, and let $f(n)$ denote the last nonzero digit in the decimal expansion of $n!$ For instance, $f(5) = 2$.

(a) Show that if $a_1, a_2, \ldots, a_k$ are *distinct* nonnegative integers, then $f(5^{a_1} + 5^{a_2} + \ldots + 5^{a_k})$ depends only on the sum $a_1 + a_2 + \ldots + a_k$.

(b) Assuming part (a), we can define $g(s) = (5^{a_1} + 5^{a_2} + \ldots + 5^{a_k})$, where $s = a_1 + a_2 + \ldots + a_k$. Find the least positive integer p for which $g(s) = g(s + p)$, for all $s \ge 1$, or else show that no such p exists.

Sol. All congruences are modulo 10.

Lemma. $f(5n) \equiv 2^n f(n)$.

Proof. We have $(5n)!$ equals

$$10^n n! \prod_{i=0}^{n-1} \frac{(5i+1)(5i+2)(5i+3)(5i+4)}{2}. \quad (*)$$

If i is even, then

$$\frac{1}{2}(5i+1)(5i+2)(5i+3)(5i+4) \equiv \frac{1\cdot 2\cdot 3\cdot 4}{2} \equiv 2,$$

and if i is odd, then

$$\frac{1}{2}(5i+1)(5i+2)(5i+3)(5i+4) \equiv \frac{6\cdot 7\cdot 8\cdot 9}{2} \equiv 2.$$

Thus the entire product above is congruent to 2^n. From (*) it is clear that the largest power of 10 dividing $(5n)!$ is the same as the largest power of 10 dividing $10^n n!$, and the proof follows.

We now show by induction on $5^{a_1} + \ldots + 5^{a_k}$ that $f(5^{a_1} + \ldots + 5^{a_k}) \equiv 2^{a_1 + \ldots + a_k}$ (which depends only on $a_1 + \ldots + a_k$ as desired).

This is true for $5^{a_1} + \ldots + 5^{a_k} = 1$, since $f(5^0) \equiv 2^0 \equiv 1$.

Case 1. All $a_i > 0$. By the lemma and induction,

$$f(5^{a_1} + \ldots + 5^{a_k}) \equiv$$

$$2^{5^{a_1-1}+\ldots+5^{a_k-1}} \cdot f(5^{a_1-1}+\ldots+5^{a_k-1})$$

$$\equiv 2^k \cdot 2^{(a_1-1)+\cdots+(a_k-1)}$$

(since $2^{5^i} \equiv 2$ for $i \ge 0$)

$$\equiv 2^{a_1+\ldots+a_k}.$$

Case 2. Some $a_i = 0$, say $a_1 = 0$. Now $(1 + 5m)! = (1 + 5m)(5m)!$, so $f(1 + 5m) \equiv (1 + 5m) f(5m)$. But $f(5m)$ is even for $m \geq 1$ since $(5m)!$ is divisible by a higher power of 2 than of 5. But $(1 + 5m)\cdot(2j) \equiv 2j$, so $f(1 + 5m) \equiv f(5m)$. Letting $m = 5^{a_2 - 1} + \ldots + 5^{a_k - 1}$, the proof follows by induction.

(b) The least $p \geq 1$ for which $2^{s+p} \equiv 2^2$ for all $s \geq 1$ is $p = 4$.

B-1. Let n be a positive integer, and define $f(n) = 1! + 2! + \ldots + n!$. Find polynomials $P(x)$ and $Q(x)$ such that

$$f(n + 2) = P(n)\ f(n + 1) + Q(n)\ f(n),$$

for all $n \geq 1$.

Sol. Since $f(n+2) - f(n+1) = (n+2)! = (n+2)(n+1)! = (n+2)[f(n+1) - f(n)]$, it follows that we can take $P(x) = x+3$ and $Q(x) = -x - 2$.

B-2. Find the minimum value of

$$(u - v)^2 + \left(\sqrt{2 - u^2} - \frac{9}{v} \right)^2$$

for $0 < u < \sqrt{2}$ and $v > 0$.

Sol. The problem asks for the minimum distance between the quarter of the circle $x^2 + y^2 = 2$ in the open first quadrant and the half of the hyperbola $xy = 9$ in that quadrant. Since the tangents to the respective curves at (1,1) and (3,3) separate the curves and are both perpendicular to $x = y$, the minimum distance is 8.

The problem can also be done by using the standard techniques of multivariable calculus.

B-3. Prove or disprove the following statement. If F is a finite set with two or more elements, then there exists a binary operation $*$ on F such that for all x,y,z in F,

(i) $x * z = y * z$ implies $x = y$ (right cancellation holds), and

(ii) $x * (y * z) \neq (x * y) * z$ (no case of associativity holds).

Sol. The statement is true. Take $x_i * x_j = x_{i+1}$, where $x_1, \ldots, x_n$ are the elements of F, $x_{n+1} = x_1$.

B-4. Find, with proof, all real-valued functions $y = g(x)$ defined and continuous on $[0, \infty)$, positive on $(0, \infty)$, such that for all $x > 0$ the y-coordinate of the centroid of the region

$$R_x = \{(s, t) \mid 0 \leq s \leq x,\ 0 \leq t \leq g(s)\}$$

is the same as the average value of g on $[0, x]$.

Sol. Such a function must satisfy

$$\frac{\int_0^x \frac{1}{2} g^2(t)\ dt}{\int_0^x g(t)\ dt} = \frac{1}{x}\int_0^x g(t)\ dt .$$

Let $z(x) = \int_0^x g(t)\ dt$. Then $z'(x) = g(x)$ and we have

$$\int_0^x \frac{1}{2}(z')^2\ dt = \frac{z^2}{x}, \qquad x > 0.$$

Differentiating, we have

$$\frac{1}{2}(z')^2 = \frac{x\cdot 2zz' - z^2}{x^2} \qquad x > 0,$$

$$(xz' - r_1 z)(xz' - r_2 z) = 0, \qquad x > 0,$$

where $r_1 = 2 + \sqrt{2}$ and $r_2 = 2 - \sqrt{2}$.

Now x, z', and z are continuous and $z > 0$, so the last equation implies that $\frac{xz'}{z} = r$, where $r = r_1$ or $r = r_2$.

Separating variables, we have $\frac{z'}{z} = \frac{r}{x}$ and it follows that $z = C_1 x^r$, $C_1 > 0$. Differentiating, we have

$$z' = g(x) = Cx^{r-1}, \qquad C > 0.$$

But g is continuous on $[0, \infty)$ and therefore we cannot have $r = r_2$ (because $r_2 - 1 = 1 - \sqrt{2} < 0$). Thus

$$g(x) = Cx^{1 + \sqrt{2}}, \qquad C > 0,$$

and one can check that such $g(x)$ do satisfy all the conditions of the problem.

B-5. For each nonnegative integer k, let $d(k)$ denote the number of 1's in the binary expansion of k (for example, $d(0) = 0$ and $d(5) = 2$). Let m be a positive integer. Express

$$\sum_{k=0}^{2^m-1} (-1)^{d(k)} k^m$$

in the form $(-1)^m a^{f(m)} (g(m))!$, where a is an integer and f and g are polynomials.

Sol. Let D_s denote the set of integers k between 0 and 2^m-1 for which $d(k) = s$. For $k \in D_s$, write $k = 2^{i_1} + \ldots + 2^{i_s}$, $i_1 < i_2 < \ldots < i_s$. We have

$$\sum_{k=0}^{2^m-1} (-1)^{d(k)} k^m$$

$$= \sum_{s=0}^{m} (-1)^s \sum_{k \in D_s} (2^{i_1} + \ldots + 2^{i_s})^m$$

$$= \sum_{s=0}^{m} (-1)^s \sum_{k \in D_s} \sum_{k_1+\ldots+k_s=m}$$

$$\frac{m!}{k_1!\ldots k_s!} (2^{i_1})^{k_1} \ldots (2^{i_s})^{k_s} \quad (*)$$

Let $j_1,\ldots,j_r$ be an r-tuple summing to m with $j_t > 0$. Then for $0 \le i'_1 < \ldots < i'_r < m$, $(2^{i'_1})^{j_1} \ldots (2^{i'_r})^{j_r}$ will occur formally in (*) whenever $(k_1,\ldots,k_s)$ consists of $(j_1,\ldots,j_r)$ with zeros thrown in. For $s \ge r$ this can be done in $\binom{m-r}{s-r}$ ways. Hence $(2^{i'_1})^{j_1} \ldots (2^{i'_r})^{j_r}$ occurs in (*) with coefficient

$$\sum_{s=r}^{m} (-1)^s \binom{m-r}{s-r} \frac{m!}{j_1!\ldots j_r!}$$

Unless $r = m$, this sum vanishes, leaving

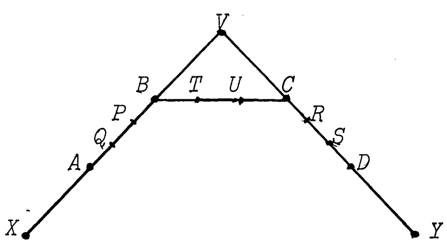
$$(*) = (-1)^m \frac{m!}{(1!)^m} 2^0 \cdot 2^1 \cdots 2^{m-1}$$

$$= m!\, 2^{\frac{m(m-1)}{2}} (-1)^m.$$

Thus, in the problem, $a = 2$, $f(x) = \frac{1}{2}x^2 - \frac{1}{2}x$, and $g(x) = x$.

B-6. A sequence of convex polygons $\{P_n\}$, $n \ge 0$, is defined inductively as follows. P_0 is an equilateral triangle with sides of length 1. Once P_n has been determined, its sides are trisected; the vertices of P_{n+1} are the interior trisection points of the sides of P_n. Thus, P_{n+1} is obtained by cutting corners off P_n, and P_n has $3 \cdot 2^n$ sides. (P_1 is a regular hexagon with sides of length 1/3.) Express $\lim_{n\to\infty} \text{Area } P_n$ in the form $\sqrt{a}/b$, where a and b are positive integers.

Sol. Suppose that XV and VY are consecutive edges in P_n and let XV be trisected by A and B, let VY be trisected by C and D. Let AB be trisected by Q and P, BC by T and U, CD by R and S (see figure).



Note that Area $\Delta PBT = \frac{1}{9}$ Area ΔBVC. Thus, the amount removed in going from P_{n+1} to P_{n+2} is 2/9 times the amount removed in going from P_n to P_{n+1}.

Since 1/3 of the original area is removed at the first step, the amount removed altogether is

$$\frac{1}{3}[1 + (2/9) + (2/9)^2 + \ldots] = \frac{1}{3}\cdot\frac{9}{7} = \frac{3}{7}$$

of the original area. Since the original area is $\frac{\sqrt{3}}{4}$, we have

$$\lim_{n\to\infty} \text{Area} = \frac{4}{7}\cdot\frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{7}.$$

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by Leon Bowden and M.M. Schiffer
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This book is based on a series of lectures given over a period of years to high school mathematics teachers. The lectures focused on topics in mechanics, questions of population growth, probability and other uses of exponential functions, optics and application of matrices to relativity theory.

The aim of the book is to illustrate the power and elegance of mathematical reasoning in science with some examples ranging from the work of Archimedes to that of Einstein. The book starts with problems of the lever, the mirror and the growth of populations and ends up with problems of space travel and atomic energy.

This excellent book will appeal to the advanced high school student, to the undergraduate and to their teachers. A sampling of the Table of Contents should inspire you to order your copy now.

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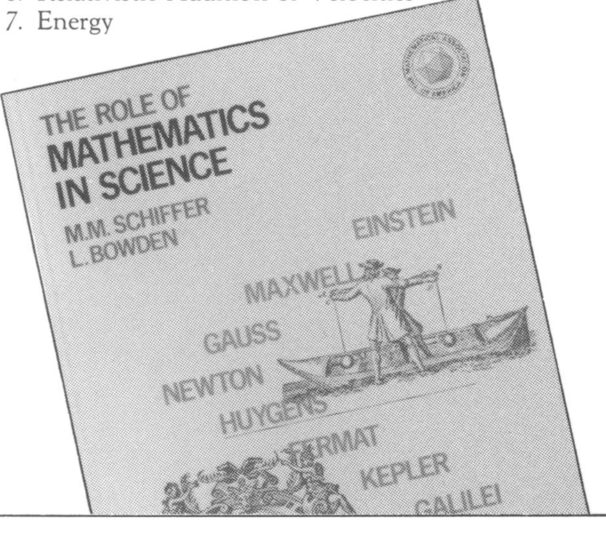

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